SECOND YEAR JUNIOR COLLEGE

MATHEMATICS

PART II

A. R. CHANDEKAR
V. D. DESHPANDE
S. V. KAMAT

S. Y. J. C. (XII Standard)
MATHEMATICS
PART II (PAPER II)

PLEDGE

"India is my country, all Indians are my brothers and sisters.

I love my country and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders respect and treat everyone with courtesy.

To my country and my people I pledge my devotion. In their well-being and prosperity alone lies my happiness."

As per the clarified syllabus of the Maharashtra & Goa State Boards of Higher Secondary Education.

S. Y. J. C. (XII Standard)
MATHEMATICS
PART II (PAPER II)
CALCULUS

By

A. R. CHANDEKAR
M.Sc. (Maths.) M.Sc. (Stats.)
Head, Department of Mathematics,
Siddharth College of Arts & Science
Bombay-400001

V. D. DESHPANDE
M. Sc. LL.B.
Head, Department of Mathematics
Wilson College,
Bombay-400007

S. V. KAMAT
M. Sc.
Dhempe College,
Panaji-Goa.

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Preface

In 1976 the S.Y.J C. (XIIth Std) syllabus was introduced for the first time. The first batch of students appeared for the S.Y.J.C. examination in 1977. The question papers set at this examination were a radical departure from the previous Inter Science examination. The questions were more of an objective nature. No proofs of theorems were asked. In general stress on theory was much less. More emphasis was laid on applications *i.e.*, on solving problems. This experiment was continued for three years. The result was a tendency towards training students in mechanical applications of theorems and formula rather than understanding and appreciating mathematical reasoning. Consequently when these students took up Mathematics in F.Y.B.Sc., they found it extremely difficult to cope up with mathematical rigour and appreciate real mathematics.

Fortunately the H.S.C. Board has done some rethinking in the matter and has issued some clarifications of the syllabus. These clearly seem to be intended to reverse the current trend. Now more emphasis is being gradually shifted to theory. Questions on theory, proofs of theorems will now be asked. There will be less number of objective questions. There will be only internal options in questions. Otherwise all questions will be compulsory. This means that the student will now have to study the theory carefully and be prepared to answer questions which will really test his understanding of the subject and concepts. He must be able to prove theorems and also apply them to problems.

An immediate result is that books based on unclarified syllabus will not now meet the requirement of the students. Further the experience in teaching for the last four ears has brought home to the authors the lacuna in the books available and also the real needs in the light of the clarified syllabus. **This is the reason why we have ventured to bring out these two volumes.**

We trust that these books will be appreciated by teachers and students. Any suggestions for improvement will be gratefully accepted.

We sincerely thank Shri T.N. Goel and Shri P.J. Singh of S. Chand & Co. for bringing out these books with their usual enthusiasm.

June 1980.

A.R. CHANDEKAR.
V.D. DESHPANDE.
S.V. KAMAT.

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Syllabus

S.Y.J.C. (MATHEMATICS) PAPER II
Calculus

1. **Convergence.** Bounded and unbounded sets of real numbers. Least upper bound and greatest lower bound. The axiom of least upper bound. Every non-empty set of real numbers which is bounded above has a least upper bound.

Convergence of a sequence to a limit. Motivation of (ϵ,N) definition. Examples of non-convergent sequences. Algebra of limits of sequences (statements only). Monotonic bounded sequence has a limit. e as a limit of the sequene $\left(1+\frac{1}{n}\right)^n$ computation of limits simple examples.

Convergence of series-comparison and ratio tests.

2. **Limits and Continuity.** Intervals, neighbourhood of a point. Deleted neighbourhood of a point. Behaviour of a function in a neighbourhood and at a point. Increasing and decreasing functions. Limit of a function at a point. Motivation of (ϵ,δ) and (ϵ,K) definitions—simple illustrative problems

Algebra of limits—(without proof)—Assume $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$

Continuity at a point and in a interval. Examples of discontinuity. Algebra of continuous functions—(statements only).

3. **Differentiation.** Derivation of a function at a point. Algebra of derivatives. Derivative of composite functions (upto two functions), chain rule. Logarithmic Differentiation. Derivative of implicit functions. Derivative of inverse functions. Geometrical and mechanical and mechanical applications of derivatives, Subtangent and subnormal. Interpretation of sign of a derivative. Related of rate and error problems. Mean value (Lagranges') theorem. Maxima and minima.

4. Integration. Integral as a lim of a sum. Integration of standard forms. Integration by substitution. Integration by parts. Integration by partial fractions.

If $$F(x)=\int f(x)\,dx$$

then $$F(b)-F(a)=\int_a^b f(x)\,dx$$

Reduction formulae for $\int_0^{\pi/2}\sin^n x\,dx$ and

$$\int_0^{\pi/2}\cos^n x\,dx.$$

Evaluation of area by definite integrals (intuitive approach). Volume of surfaces of revolution.

5. Differential Equations. Graph of $\sin(ax+b)$, $\cos(ax+b)$. Formation of a differential equation. First order differential equation in which variables are separable.

Second order differential equation of simple harmonic motion.

$$\frac{d^2x}{dt}+k^2x=0$$

Graph of $y=a\sin(bx+c)$

and $y=a\cos(bx+c)$.

Application to problems on motion under gravity, growth and decay and law of cooling.

CLARIFICATIONS

CALCULUS—PAPER II

Sr, No.	Unit/Subunit	Scope, limitations and concepts to be emphasized	Suggestions
1	2	3	4
		SECTION I	
1.	**Convergence**		
1.	Bounded sets	The definition of upper bound, least upper bound, lower bound, greatest lower bound	Review of sequences. Graphical representation must be emphasized throughout.
		(1) Condition that a given number is *not* a lower bound/upper bound.	

Sr. No.	Unit/Subunit	Scope, limitations and concepts to be emphasized	Suggestions
1.	2.	3.	4
		(2) not least upper bound/greatest lower bound. Axiom of upper bound. Corresponding property of the greatest lower bound.	
2.	Limit	Limit of a sequence. Motivation of $(\in, -N)$ definitions. Convergent and nonconvergent sequences. Examples of nonconvergent sequences. Algebra of limits of sequences (statements only) Computation of limits. Uniqueness of a limit of a sequence. Axiom : Monotonic bounded sequences are convergent. Discussion of convergence of geometric sequences, considering different cases. The sequence $S_n=\left(1+\frac{1}{n}\right)^n$ is monotonic and bounded must be shown and is expected from the students.	$\lim_{n\to\infty}\frac{1}{n}=0$ using the $(\in -N)$ definition is to be shown in the class. While computing limits, the steps must clearly indicate the laws of limits which have been used in each step.
3.	Convergence of Series	Concept of partial sums. Convergence of series as convergence of a sequence of partial sums. Convergence of $\sum\frac{1}{n^p}$ expected from students. Derivation of comparison and ratio tests in the limit from to be shown in the class. Applications only expected from the students.	Finding the sum of an infinite-series is not expected.

1	2	3	4
2.	**Limits and Continuity**		
1.	Neighbourhood	Intervals, neighbourhood, deleted neighbourhood. Behaviour of a function in a neighbourhood and at a point. $f(x)$ is said to increasing at $x=a$ if for some interval $(a-\delta,\ a+\delta)$, and for any two points x_1 , x_2 in this interval such that $x_1>x_2$ we have $f(x_1)>f(x_2)$ Similar definitions for decreasing, strictly decreasing & strictly decreasing functions.	$\lim_{x\to\infty} f(x)$, $\lim_{x\to a} f(x)=\infty$ are not expected. In examples of finding the limits of sequences and functions, separate steps must indicate clearly the theorems on limits which are being used, but the theorems need not be stated verbatim.
2.	Limit of a Function	Motivation of $(\epsilon,, \delta)$ definitions. Algebra of limits (without proof). $\lim_{x\to 0} f[q(x)]=f\left[\lim_{x\to a} q(x)\right]$ $\lim_{x\to a} \frac{\sin x}{x}=1$ and $\lim_{x\to 0} \cos x=1$ to be assumed. No other trigonometric limit should be assumed. $\lim_{x\to 0} \frac{a^x-1}{x}=\log_e a$ to be accepted (without proof) and all other steps should be shown clearly. Uniqueness of a limit of a function.	

1	2	3	4
3. Continuity		Continuity at a point and in an interval to be defined. Examples of discontinuity. Algebra of continuous functions (statements only). Informal dicussion of continuity of standard functions, viz., constant function, power function, polynomial function, rational function, trigonometric functions, exponential function, logarithmic function. Discussion of inverse trigonometric functions through graphs.	Entire motivation of continuity and discontinuity should be through graphs.
3. Differentiation			
1. Derivatives		Definition of a derivative of a function f as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, $(h\neq 0)$ and $\operatorname{Lim}_{x\to a} \frac{f(x)-f(a)}{x-a}$ Using the definition to find the derivatives of : x^n, n positive integer ; trigonometric functions ; exponential functions (e^x, a^x) ; logarithmic function. Proofs of theorems on differentiation : $u+v$; ku uv, u/v, composite function (use Leibnitz' notation). Derivatives of inverse trigonometric functions; Derivatives of implicit functions, parametric differentiation ; logarithmic differentiation. Concept of higher order derivatives.	
2. Applications of Derivatives		Geometrical interpretation of the derivative at a point. Equations of tangent and normal to a curve at a point. Subtangent, subormal.	

1	2	3	4
		Angle between two curves. Orthogonality of two curves.	
		Interpretation of the sign of the derivative, to be shown informally, using the definition of increasing and decreasing functions. Applications to polynomial functions only.	
		Maxima and minima of functions and their applications. To decide the maxima and minima of a function, solve $f'(x)=0$. If $f'(c)=0$, use the definition to decide the maximum or minimum. *Test of second order derivative is not permitted.* Lagrange's Mean Value theorem, (Geometrical demonstration).	Define maxima and minima geometrically as : A function f has a maximum at $x=c$ if in some nbd of $x=c$ it increases from left upto c, and decreases to the right velocity of $x=c$. Similarly for minimum.
		Derivative as a rate measure. Related rate problems. velocity and acceleration. Errors and approximations, percentage error.	
4. Integration			
1. Antiderivative		Integral as antiderivative. Rules of integration : $\int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx$; $\int k.f(x)\,dx = k\int f(x)\,dx$; to be proved. Integration of standard forms. Rule of integration by substitution and parts to be proved. Integration by partial fractions (denominators involved should be factorisable into linear and distinct factors).	
2. Definite integrals		Notion of an area bounded by curve of a continuous function $y=f(x)$, ordinates at $x=a$, $x=b$ and the x-axis as the limit of a sum. Integral of a continuous function $y=f(x)$	

1	2	3	4
		defined on $[a, b]$ as an area bounded by the curve $y = f(x)$, ordinates at $x = a$, $x = b$ and the x-axis. Examples on definite integrals as a limit of a sum. Fundamental theorem of integral calculus (without proof) properties and evaluation of definite integrals.	Converse problems not expected.
		$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$; $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is an even function and $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd.	Proofs of properties expected.
		Reduction formulae for $\int_0^{\pi/2} \sin^n n\,dx$, $\int_0^{\pi/2} \cos^n x\,dx$. In any numerical problem the first reduction must be shown by the method of integration by parts. Other steps can be induced. For example— $\int_0^{\pi/2} \sin 7x\,dx = \frac{6}{7}\int_0^{\pi/2} \sin^5 x\,dx$ has to be calculated and shown and subsequent steps could be written down by induction.	
	3. Applications of Integration	Evaluation of area under a curve bounded by the axis of x and between the two ordinates ;	
		Evaluation of areas enclosed between two curves, (simple examples) when points of intersection are given.	
		Volumes of solids of revolution (about the axes of coordinates only) as a sum of volumes of elemental discs expressed as definite integrals.	

Sr. No.	Unit/ Subunit	Scope, limitations and concepts to be emphasized	Suggestions
1	2	3	4
5.	**Differential Equations**		
	1. Formation of Differential Equations	Formation of differential equations of first and second order of a family of curves elimination of parameters. Solution of a first order differential equation in which variables are separrable, and reducible to variables separable where a substitution is given.	
	2. Simple Harmonic Motion	Formation of $\frac{d^2x}{dt^2} + k^2x = 0$, as a linear motion of a particle being attracted towards a fixed point, the force of attraction varying as its distance from the fixed point. Solution of this differential equation in the form $x = A \cos kt + B \sin kt$ or $x = R \sin (\alpha + kt)$, Graph of $x = R \sin (\alpha + kt)$.	
	3. Application	Application to problems on simple harmonic motion, motion under gravity, growth and decay and the law of cooling.	Sufficient drilling must be given in conversion of simple physical situation into differential equations depending on the forms discussed above.

1

Sequences and Series

1·1. Sequences

A sequence is a function whose domain is the set of all natural numbers or positive integers.

If $f : \mathbb{N} \to \mathbb{R}$ is a sequence function, then its values are $f(1)$, $f(2)$, $f(3)$, ..., $f(n)$..... These terms are respectively called the first term, second term, ..., nth term of the sequence, and more often than not we are concerned with the values of the sequences or *the range of the sequence function*. We normally write the terms of the sequence as $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ etc., in place of $f(1)$, $f(2)$, $f(3)$, ..., $f(n)$.

a_n is called the nth term of the sequence and we shall represent the sequence as (a_n) or $<a_n>$. We shall represent different sequences as (a_n), (b_n), (c_n), ...etc.

We give below some examples of sequences.

(*i*) 7, 7, 7, 7, 7, 7, 7, 7, ...

(*ii*) 0, 0, 0, 0, 0, 0, 0, 0, ...

(*iii*) 1, 3, 5, 7, 9, 11, 13, 15, 17, ...

(*iv*) 2, 5, 10, 17, 26, 37, 50, 65, 82, 101.....

(*v*) $3, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, \ldots\ldots$

(*vi*) 100, 97, 94, 91, 88, 85, 82, 79, 76, 73, ...

(*vii*) ·1, ·11, ·111, ·1111, ·11111, ...

(*viii*) $\sqrt{7}, \quad \sqrt{7\sqrt{7}}, \quad \sqrt{7\sqrt{7\sqrt{7}}}, \quad \sqrt{7\sqrt{7\sqrt{7\sqrt{7}}}}, \ldots\ldots$

(*ix*) 1, −1, 1, −1, 1, −1, 1, −1, 1, −1, 1, −1, ...

(*x*) 1, −2, 3, −4, −5, 6, −7, 8, −9, 10, −11, 12, −13 .....

A careful scrutiny of the above sequences reveals that each sequence is controlled by some sort of formula. In the first sequence, each term is 7, and in the second sequence, each term is 0. A sequence, whose all terms are the same, is called a *constant sequence.* A sequence whose all terms are 0 is called a *null* sequence. In the (*iii*) rd sequence, we have enumerated all *odd numbers* and the nth term of the sequence is given by $(2n-1)$, In sequence (*iv*), the nth term is (n^2+1) where the nth term of sequence (*v*) is 3^n. The terms of the sequence (*v*) are in *geometric progression* with a common ratio 3. The sequence (*vi*) has its terms in *arithmetic progression* with the common difference -3. In the (*vii*) th sequence, the nth term has in the first n places after the decimal points, 1 and the other places zeros. In the same way we can put the terms of the sequence (*viii*) in some formula—which we shall discuss later. The terms of the sequence (*ix*) are alternately 1 and -1 and the nth term of this sequence is given by $(-1)^{n+1}$. Such a sequence is called an *alternating sequence.* In the sequence (*x*), the nth term is given by $(-1)^{n+1}n$.

Thus, we observe that we are dealing with sequence in which the nth term of the sequence can be obtained by some formula. It is necessary to stretch one's *imagination* and develop quick mathematical thinking to link the nth term of a sequence (a_n) with natural number n.

1·2. In the 11th standard, only arithmetic and geometric sequences were discussed in the form of *progressions.* The discussion was restricted to finding the general term (nth term) and an expression (or a formula) for the sum of the first n terms of the sequence. The number of terms of the progression used to be finite. We now discard this restriction and consider sequences which have infinitely many terms and therefore we call these sequences as infinite sequences. The discussion in this chapter, however, will be centred around the nth term of the sequence and the sum of the first n terms of the sequence. In the first part, we shall discuss the behaviour of the nth term as n increases without any ceiling. This will lead us to the concepts of convergent and non-convergent sequences. A similar consideration of the sum of the first n terms of the sequence involves the discussion of a new concept, that of infinite series.

1·3. Convergent sequences

Let (a_n) be a sequence such that,

$$a_n = 2 + \frac{(-1)^n}{n}$$

Let us write down a few terms of this sequence. We have, therefore :

$$a_1 = 2-1 = 1,$$

$$a_2 = 2+\frac{1}{2} = \frac{5}{2},$$

$$a_3 = 2-\frac{1}{3} = \frac{5}{3},$$

$$a_4 = 2+\frac{1}{4} = \frac{9}{4},$$

$$a_5 = 2-\frac{1}{5} = \frac{9}{5},$$

$$a_6 = 2+\frac{1}{6} = \frac{13}{6},$$

$$a_7 = 2-\frac{1}{7} = \frac{13}{7}, \text{ etc.}$$

Observe that the first term is < 2,
the second term is > 2,
the third term is < 2,
the fourth term is > 2 and so on.

Also each term is closer to 2 than the previous term.

The following figure makes this clear.

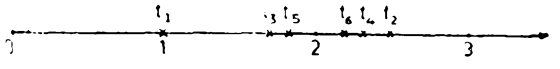

Fig. 1·1.

Now $|a_1-2| = 1$, $|a_2-2| = \frac{1}{2}$, $|a_3-2| = \frac{1}{3}$, $|a_4-2| = \frac{1}{4}$
$|a_5-2| = \frac{1}{5}$, $|a_6-2| = \frac{1}{6}$ and so on.

In general

$$|a_n-2| = \frac{1}{n}.$$

Observe that

$$|a_7-2| = \frac{1}{7} < \frac{1}{6}, \quad |a_8-2| = \frac{1}{8} < \frac{1}{6}.$$

and in general

$$|a_n-2| = \frac{1}{n} < \frac{1}{6} \text{ for all } n > 6.$$

This means that all terms after the 6th will be inside the interval.

$$\left(2-\frac{1}{6}, 2+\frac{1}{6}\right)$$

of which 2 is the mid point. Similarly all terms after the 7th will lie inside $\left(2-\frac{1}{7}, 2+\frac{1}{7}\right)$ and all terms after the 8th will lie inside $\left(2-\frac{1}{8}, 2+\frac{1}{8}\right)$.

Conversely suppose we are given the interval

$$\left(2-\frac{1}{1000}, 2+\frac{1}{1000}\right)$$

We know that

$$|a_n-2| = \frac{1}{n}$$

$$\therefore \quad |a_{1000}-2| = \frac{1}{1000}$$

$$|a_{1001}-2| = \frac{1}{1001}$$

$$|a_{1002}-2| = \frac{1}{1002} \text{ etc.}$$

$\therefore$ all terms after t_{1000} will lie inside $\left(2-\frac{1}{1000}, 2+\frac{1}{1000}\right)$

One can now verify that given any (small) interval around 2, we can always find a positive integer m such that all terms after t_m will be inside that interval. Mathematically this can be stated as : given $\epsilon>0$ we can find a positive integer m such that,

$$|a_n-2| < \epsilon \text{ for all } n \geqslant m.$$

Here the interval is $(2-\epsilon, 2+\epsilon)$. It is called the ϵ—neighbourhood of 2. To conclude the discussion we observe that,

(*i*) as we take larger and larger values of n, the terms of the sequence (a_n) where $a_n = 2+\frac{(-1)^n}{n}$ come closer and closer to 2 and,

(*ii*) given $\epsilon>0$ we can find a positive integer m such that,

$$|a_n-2| < \epsilon \text{ for all } n \geqslant m.$$

We say in this case that the sequence (a_n) converges to the limit 2 as n tends to infinity and write,

$$\lim_{n\to\infty} a_n = 2.$$

2 is called the limit of the sequence (a_n).

Ex. 1. Consider the sequence (a_n), $a_n = 3+\frac{1}{n^2}$ Let us write down a few terms of the sequence. We have,

$$a_1 = 3+1, \quad a_2 = 3+\frac{1}{4} \quad a_3 = 3+\frac{1}{9},$$

$$a_4 = 3+\frac{1}{16},\ a_5 = 3+\frac{1}{25},$$

$$a_6 = 3+\frac{1}{36},\ a_7 = 3+\frac{1}{49},\ a_8 = 3+\frac{1}{64},$$

we represent these terms in the following figure.

Fig. 1·2

It will be observed that as n becomes larger, a_n comes closer to 3.

Also we have,

$$|\ a_n - 3\ | = \frac{1}{n^2}$$

Consider the interval

$$\left(3-\frac{1}{10{,}000},\ 3+\frac{1}{10{,}000}\right)$$

Observe that

$$|\ a_{100}-3\ | = \frac{1}{100^2} = \frac{1}{10{,}000}$$

$$|\ a_{101}-3\ | = \frac{1}{101^2} < \frac{1}{100^2}$$

$$|\ a_{102}-3\ | = \frac{1}{102^2} < \frac{1}{100^2} \text{ etc.}$$

$\therefore$ for all $n > 100$.

$$|\ a_n - 3\ | < \frac{1}{100^2}$$

$\therefore$ all the terms after a_{100} will lie inside the interval,

$$\left(3-\frac{1}{10{,}000}, 3+\frac{1}{10{,}000}\right)$$

Similarly given any interval $(3-\epsilon,\ 3+\epsilon)$ we can always find a positive integer m such that all terms after a_m will lie inside this interval. Mathematically we say : given $\varepsilon > 0$ we can find a positive integer m such that,

$$|\ a_n - 3\ | < \epsilon \text{ for all } n \geqslant m.$$

$\therefore$ $$\lim_{n\to\infty} a_n = 3.$$

After carefully studying the above two examples, we are now in a position to appreciate the following definition of the limit of a sequence.

Def : 1.3.1. A sequence (a_n) is said to converge to the limit l if given $\epsilon > 0$ we can find a positive integer m, where m may depend upon ϵ, such that $|a_n - l| < \epsilon$ for all $n \geqslant m$.

We then write

$$\lim_{n \to \infty} a_n = l.$$

Illustrative Examples

Ex. 1. Show that the sequence (a_n), where $a_n = \frac{3n+1}{2n}$ converges to $\frac{3}{2}$.

Sol. We have,

$$a_n = \frac{3n+1}{2n}$$

$$= \frac{3}{2} + \frac{1}{2n}.$$

$$\therefore \left| a_n - \frac{3}{2} \right| = \left| \frac{3}{2} + \frac{1}{2n} - \frac{3}{2} \right| = \left| \frac{1}{2n} \right| = \frac{1}{2n}$$

Now, let $\epsilon > 0$ be given,

$$\therefore \left| a_m - \frac{3}{2} \right| < \epsilon \quad \text{if } \frac{1}{2n} < \epsilon$$

and consequently

$$\frac{1}{2n} < \epsilon \text{ if } n > \frac{1}{2\epsilon}.$$

Let m be a positive integer such that $m > \frac{1}{2\epsilon}$.

Clearly $n > \frac{1}{2\epsilon}$ whenever $n > m$.

$$\therefore \left| a_n - \frac{3}{2} \right| < \epsilon \text{ if } n \geqslant m.$$

$$\therefore \lim_{n \to \infty} a_n = \frac{3}{2}.$$

It may be, however, remembered that every sequence need not be convergent and consequently every sequence need not have limit

Remark. It can be proved that the limit l of a convergent sequence is unique, whenever it exists. Also, only a finite number l can qualify to become the limit. This means whenever a sequence is convergent, it has limit and the limit is *unique* and *finite*.

We shall now consider those sequences which are not convergent.

1·4. Non Convergent Sequences

1. Divergent Sequences

Consider the sequence (a_n) given by $a_n = n^2+2$. Let us write down some terms of this sequence. They are

$a_1 = 2+1^2 = 3$, $a_2 = 2+2^2 = 6$, $a_3 = 2+3^2 = 11$,

$a_4 = 2+4^2 = 18$, $a_5 = 2+5^2 = 27$, ...,

$a_{100} = 2+100^2 = 10{,}002$, $a_{1000} = 2+1000^2 = 1000{,}002$...etc.

It is clear that the terms of this sequence go on increasing with increasing values of n.

Now, consider a positive number, say 1578938·78.

Clearly $a_{1500} = 2+(1500)^2 = 2250002 > 1578938{\cdot}78$

Since the terms go on increasing with n, we have

$a_{1501} > 1578938\ 78$

$a_{1502} > 1578938{\cdot}78$ etc.

In general,

$a_n > 1578938{\cdot}78$ for all natural numbers $n \geqslant 1500$. It can be similarly verified that given any large positive number, we can always find a positive integer m such that a_m and all the terms after a_m are greater than the given number.

Mathematically we state this as given any $k>0$, we can find a positive integer m such that

$a_n > k$ for all $n \geqslant m$.

We shall now put this in formal definition.

Def. 1. A sequence (a_n) is said to *diverge* to $+\infty$, if given $k > 0$, we can find a positive integer m such that

$$a_n > k \text{ for all } n \geqslant m.$$

we then write

$$\lim_{n\to\infty} a_n = +\infty.$$

Some sequences diverge to $-\infty$. The following example illustrates this.

Ex. 1. Show that the sequence (a_n) given by $a_n = -n^3$ diverges.

Sol. $a_n = -n^3$

$\therefore$ $a_1 = -1$, $a_2 = -(2)^3 = -8$,

$a_3 = -(3)^3 = 27$. $a_4 = -(4^3) = -64$,

$a_5 = -(5)^3 = -125$, $a_{100} = -(100)^3 = -1000{,}000$

It is clear that all the terms of this sequence are negative and they go on increasing in magnitude as n increases.

Consider now a positive integer, say 2000000. Then, we observe that

$$a_{130} = -(130)^3 = -2197000 < -2000000,$$

Also, since all the terms of this sequence go on increasing in magnitude,

$$a_{131} < -2000000, \quad a_{132} < -2000000 \text{ etc.}$$

In general,

$$a_n < -2000{,}000 \text{ for all } n \geqslant 130.$$

It can be similarly verified that given any large positive number k, we can always find a positive integer m such that a_m and the terms of the sequence after a_m are less than $-k$.

Mathematically we can state this as : given $k > 0$, we can find a positive integer m such that

$$a_n < -k \text{ for all } n \geqslant m.$$

In this case, we say that the sequence (a_n) diverges to $-\infty$. This can be put in the formal definition as follows :

Def. 2. A sequence (a_n) is said to diverge to $-\infty$ if given $k > 0$, we can find a positive integer m such that

$$a_n < -k \text{ for all } n \geqslant m.$$

We then write

$$\lim_{n \to \infty} a_n = -\infty.$$

1·5. Oscillating Sequences

So far we have considered two types of sequences—convergent sequences and divergent sequences. However, there are sequences which are neither convergent nor divergent. These sequences are known as *oscillating* sequences.

Consider the sequence (a_n); $a_n = (-1)^n$. The terms of this sequence alternately become -1 and 1, irrespective of the magnitude of n. Since the terms of this sequence do not increase in magnitude with increase in value of n, this sequence *cannot* be a divergent sequence. Also, since the terms of this sequence alternately become -1 and 1, no real number can serve as the *limit* of this sequence and therefore this sequence is *not* a convergent sequence. Consequently, this sequence is an oscillating sequence.

The sequence (a_n); $a_n = (-1)^n n$, takes the values -1, 2, -3, 4, -5, 6, -7, ... The terms of this sequence are alternately positive and negative. The odd terms of the sequence diverge to $-\infty$ and the even terms of this sequence diverge to $-\infty$. It is evident that this sequence as a whole, does not diverge to $-\infty$ or to $+\infty$.

Also, it is clear that this sequence does not converge. Therefore, this is an oscillating sequence. We say that the sequence in this case oscillates infinitely.

In the last two examples, we have considered sequences which are alternately positive and negative to illustrate oscillating sequences. However, an oscillating sequence need not be an alternating sequence as shown by the following example.

Consider the sequence (a_n) given by $a_n = 2+(-1)^n$.

Here we observe that

$a_1 = 1,\ a_2 = 3,\ a_3 = 1,\ a_4 = 3, a_5 = 1, a_6 = 3$ .....

Thus, the terms of the sequence are alternately 1 and 3 respectively.

Since the terms of this sequence do not increase or decrease progressively, this sequence is *not* divergent. It can be verified that no real number can be a limit of this sequence and hence this sequence is not convergent. Therefore, this sequence is an oscillating sequence.

We shall now give formal definitions :

Def. 1. A non-convergent and non divergent sequence (a_n) is said to be finitely oscillating if there exists a positive number k such that

$$| a_n | \leqslant k \text{ for all natural numbers } n.$$

Def. 2. A non-convergent and non-divergent sequence (a_n) is said to be infinitely oscillating if it is not finitely oscillating.

EXERCISE 1·1.

1. Show that the sequences whose nth terms are given below, are finitely oscillating :

(i) $(-1)^{n-1}\,2$ (ii) $\sin\frac{n\pi}{2}$ (iii) $\cos n\pi$

(iv) $\sin[(2n-1)\pi/2]$ (v) $5+(-1)^n 2$ (vi) $1+(-1)^n$.

2. Show that the sequences, whose nth terms are given below, are in infinitely oscillating :

(i) $(-1)^n\,2^n$ (ii) $n\sin\left(\frac{n\pi}{2}\right)$ (iii) $n^2\cos n\pi$

(iv) $n\cos\left(\frac{n\pi}{2}\right)$ (v) $(-1)^n n^5$ (vi) $n^3\tan\left[\frac{(-1)^n\pi}{4}\right]$

(vii) $-n\sec(n\pi)$.

1·6. Algebra of Limits of Sequences

On the basis of above discussions, we conclude that a sequence is either convergent or non-convergent. If it is non-convergent, then it may be divergent (diverging to $+\infty$ or $-\infty$) or oscillating (oscillating finitely or infinitely). If a sequence has a limit, then it is

convergent. Whenever the limit of a sequence exists, it is *unique* and *finite*. *If* a sequence does not have a limit, then it is non-convergent. Thus, the problem of deciding whether a sequence is convergent or divergent reduces to the problem of finding its limit, if it exists.

To find the limits of convergent sequences, we shall depend on some theorems, (or results) and a few standard sequences. These results are exactly similar to the results of finding the limits of functions which are discussed in the XIth standard.

Theorems on limits of Sequences

Let (a_n) and (b_n) be two convergent sequences such that

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b$$

Then we can prove the following theorems :

Thm. 1. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

$= a + b.$

Thm. 2. $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

$= a - b.$

Thm. 3. $\lim_{n\to\infty} (a_n . b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

$= a . b.$

Thm. 4. $\lim_{n\to\infty} (k . a_n) = k \lim_{n\to\infty} a_n = k . a$

Thm. 5. If $b_n \neq 0$ for all n and $b \neq 0$ then $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right)$

$$= \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{a}{b}.$$

Thm. 6. (*i*) $\lim_{n\to\infty} \text{Sin } (a_n) = \text{Sin } (\lim_{n\to\infty} a_n)$

$= \text{Sin } a$

(*ii*) $\lim_{n\to\infty} \text{Cos } (a_n) = \text{Cos } (\lim_{n\to\infty} a_n)$

$= \text{Cos } a$

(*iii*) $\lim_{n\to\infty} (x^{a_n}) = x^{(\lim_{n\to\infty} a_n)} = x^a,$

provided $x > 0$.

(*iv*) $\lim_{n\to\infty} \log (a_n) = \log (\lim_{n\to\infty} a_n)$

$= \log a,$

provided $a_n > 0$ for all n and $a > 0$.

(*vi*) If $a_n = k$ for all n, then $\lim_{n\to\infty} a_n = k$.

Thm. 7. (*i*) If $a_n \leqslant b_n$ for all n, then

$$\lim_{n\to\infty} a_n \leqslant \lim_{n\to\infty} b_n;\ i.e.,\ a \leqslant b.$$

(*ii*) If $a_n < b_n$ for all n, then

$$\lim_{n\to\infty} a_n \leqslant \lim_{n\to\infty} b_n\ ;\ i.e.,\ a \leqslant b.$$

(*iii*) $\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right| = |a|$.

The above theorems imply that whenever (a_n) and (b_n) are convergent sequences, then (a_n+b_n), (a_n-b_n), $(k\,a_n)$, $(a_n\,b_n)$, (a_n/b_n) $(\sin a_n)$, $(\cos a_n)$, $(\log a_n)$, (x^{a_n}), $(|a_n|)$ are all convergent sequences and they converge to the limits indicated above. Therefore, whenever we want to discuss the convergence of a sequence, we shall first explore the possibility of expressing the given sequence in terms of a sum or difference or product or ratio etc., of two or more convergent sequences. If we succeed in doing so, with the help of above mentioned theorems and some standard results given below, we can obtain the limit of the given sequence without any difficulty. Before we actually deal with examples directly, we shall consider some standard results.

1·7. Some Standard Sequences

1. Let (a_n) be given by $a_n = n^p$. Then,

(*i*) (a_n) converges to zero if $p < 0$, and (*ii*) (a_n) diverges to $+\infty$ if $p > 0$.

2. Let (a_n) be a sequence given by $a_n = x^n$.

Then the sequence (a_n)

(*a*) converges for $-1 < x < 1$ and $\lim_{n\to\infty} a_n = 0$,

(*b*) converges when $x = 1$ and $\lim_{n\to\infty} a_n = 1$,

(*c*) diverges to $+\infty$ if $x > 1$,

(*d*) oscillates finitely if $x = -1$; and

(*e*) oscillates infinitely if $x < -1$.

Proof. (*a*) Let $|x| < 1$ or $-1 < x < 1$ and $x \neq 0$.

Now, $|a_n| = |x^n| = |x|^n$.

Let $\epsilon > 0$ be given.

Then, $|a_n| < \epsilon$ if $|x^n| < \epsilon$.

But, $|x^n| < \epsilon$ if $n \log |x| < \log \epsilon$.

Since $|x| < 1$, $x \neq 0$ and ϵ is supposed to be small, $\log |x| < 0$ and $\log \epsilon < 0$.

$$\therefore \quad n \log |x| < \log \epsilon \text{ if } n > \frac{\log \epsilon}{\log |x|}.$$

Choosing m a positive integer such that $m > \frac{\log \epsilon}{\log |x|}$, we observe that $|a_n - 0| = |a_n| < \epsilon$ if $n \geqslant m$.

But this result is true for all small ϵ.

$$\therefore \quad \lim_{n \to \infty} a_n = 0$$

Also, if $x = 0$ then $|x| = 0$ and hence $a_n = 0$ for all n. This being a constant sequence $\lim_{n \to \infty} a_n = 0$.

$$\therefore \quad \lim_{n \to \infty} a_n = 0 \text{ for } |x| < 1.$$

$\therefore$ If $|x| < 1$, the sequence converges to 0.

(*b*) Let $x = 1$.

Then $a_n = (x^n) = 1$ for all n.

$$\therefore \quad |a_n - 1| = |1 - 1| = 0 < \epsilon \text{ for all } n.$$

$$\therefore \quad \lim_{n \to \infty} a_n = 1.$$

$\therefore$ The sequence converges to 1 when $x = 1$.

(*c*) Let $x > 1$.

Clearly $a_n = x^n > 0$ for all n.

Let $k > 0$ be any (large) number given. Then $a_n > k$ whenever $x^n > k$.

Now, $x^n > k$ whenever $n \log x > \log k$.

Since $x > 1$ and k is large, without any loss of generality, we can assume that $\log x$ and $\log k$ are both positive.

$\therefore$ $n \log x > \log k$ if $n > \frac{\log k}{\log x}$. Choose a positive integer m such that $m > \frac{\log k}{\log x}$.

Then, $a_n > k$ for all $n \geqslant m$.

$\therefore$ given any large positive number k, we can choose a positive integer m such that $a_n > k$ for all $n \geqslant m$.

$\therefore$ The sequence (a_n) diverges to $+\infty$.

(*d*) Let $x = -1$.

$$\therefore \quad a_n = x^n = (-1)^n.$$

The terms of this sequence are namely $-1, 1, -1, 1, -1, 1, -1, 1,$ ..... *i.e.*, the terms are alternately -1 and 1.

Clearly the sequence is not a convergent sequence. But $| a_n | = 1$. Therefore the sequence is not divergent. Hence the sequence oscillates finitely.

(*e*) Let $x < -1$ and let $| x | = a$.

Clearly $x = -a$ and $a > 1$.

$\therefore \quad a_n = x^n = (-a)^n = (-1)^n a^n \; ; \; a > 1.$

It is evident that the terms of the sequence are alternately and positive.

Let k be any large positive number.

Now, $| a_n | = | (-1)^n a^n | = a^n$.

$\therefore \quad | a_n | > k$ if $a^n > k$.

$\therefore \quad | a_n | > k$ if $n \log a > \log k$.

Since $a > 1$ and k is large, both $\log a$ and $\log k$ are positive.

Choosing a positive integer m such that $m > \dfrac{\log k}{\log a}$,

We observe that

$| a_n | > k$ if $n \geqslant m$.

$\therefore$ The sequence $(| a_n |)$ is divergent to $+\infty$.

$\therefore$ The sequence (a_n) cannot be convergent otherwise $(| a_n |)$ would have been a convergent sequence.

Also, the terms of the sequence are alternately negative and positive. Therefore the sequence (a_n) is neither divergent to $-\infty$ nor divergent to $+\infty$.

$\therefore$ The sequence (a_n) is an infinitely oscillating sequence.

3. Let (a_n) be a sequence where

$$a_n = a^{1/n} \; ; \; a > 0.$$

Then (a_n) converges to **1.**

Proof. (*i*) Suppose that $a > 1$.

$\therefore \quad a^{1/n} > 1$ for all n.

Put $a^{1/n} = 1 + h$. Clearly $h > 0$.

$\therefore \quad a = (1+h)^n$

$$= 1 + nh + \frac{n(n-1)}{2!} h^2 + \frac{n(n-1)(n-2)}{3!} h^3 \ldots\ldots + h^n,$$

$$> 1 + nh.$$

$\therefore \quad a > 1 + nh.$

$\therefore \quad h < \dfrac{a-1}{n}$.

Also, $| a^{1/n} - 1 | = a^{1/n} - 1 = h$.

Now, let $\epsilon > 0$ be given.

$\therefore \quad | a^{1/n} - 1 | < \epsilon$ if $h < \epsilon$

$\therefore \quad | a^{1/n} - 1 | < \epsilon$ if $\dfrac{a-1}{n} < \epsilon$.

Choose a positive integer m such that $m > \dfrac{a-1}{\epsilon}$.

Then, we have

$| a^{1/n} - 1 | < \epsilon$ whenever $n \geqslant m$.

$\therefore \quad \lim_{n \to \infty} a^{1/n} = 1.$

(*ii*) Suppose now that $0 < a < 1$.

Put $b = \dfrac{1}{a}$.

Clearly $b > 1$ and $a = \dfrac{1}{b}$.

$\therefore \quad \lim_{n \to \infty} (b^{1/n}) = 1$ by (*i*) above.

But $a^{1/n} = \dfrac{1}{b^{1/n}}$.

$\therefore$ By Thm. 5 and Thm. 6 (*v*), we have

$$\lim_{n \to \infty} (a^{1/n}) = \lim_{n \to \infty} \left(\frac{1}{b^{1/n}} \right)$$

$$= \frac{\lim_{n \to \infty} 1}{\lim_{n \to \infty} b^{1/n}} = \frac{1}{1} = 1.$$

$\therefore \quad \lim_{n \to \infty} a^{1/n} = 1$

(*iii*) Let $a = 1$

Then $a_n = a^{1/n} = 1$ for all n.

$\therefore$ By Thm. 6 (*v*).

$\lim_{n \to \infty} a_n = 1.$

By (*i*), (*ii*) and (*iii*) above,

$\lim_{n \to \infty} a^{1/n} = 1$ if $a > 0$.

4. The sequence (a_n) given by $a_n = (-1)^n n^p$.

(*i*) converges to 0 if $p < 0$ and

(*ii*) oscillates if $p \geqslant 0$.

Illustrative Examples

Ex. 1. Examine the convergence of the sequences whose nth terms are given below.

(*i*) n^2, (*ii*) n^{-3} (*iii*) $(-\frac{1}{10})^n$; (*iv*) $(\frac{2}{3})^n$; (*v*) $(\frac{3}{2})^n$; (*vi*) $(\sqrt[n]{5})$; (*vii*) $(-1)^n n$.

Sol. (*i*) Put $a_n = n^2$. Then (a_n) is of the type (n^p) where $p = 2 > 0$.

$\therefore$ The sequence diverges to $+\infty$.

(*ii*) Put $a_n = n^{-3}$. This sequence (a_n) is of the type (n^p) where $p = -3 < 0$.

Clearly the sequence (a_n) converges to 0.

(*iii*) Put $a_n = (-\frac{1}{10})^n$. In this case, the sequence of the type (x^n) where $x = -\frac{1}{10}$. Since $-1 < x < 1$, the sequence converges to 0.

(*iv*) Put $a_n = (\frac{2}{3})^n$. Then $a_n = x^n = (\frac{2}{3})^n$ and $0 < x < 1$.

$\therefore$ The sequence (a_n) converges to 0.

(*v*) Put $a_n = (\frac{3}{2})^n$. Once again, here $a_n = (\frac{3}{2})^n$ and $\frac{3}{2} > 1$. The sequence is of the type (x^n) where $x = \frac{3}{2} > 1$.

Clearly the sequence (a_n) diverges to $+\infty$.

(*vi*) Put $a_n = \sqrt[n]{5} = 5^{1/n}$.

There $a_n = a^{1/n}$ where $a = 5 > 1$.

$\therefore$ The sequence (a_n) converges to 1.

(*vii*) Put $a_n = (-1)^n n$.

We have, then $a_n = (-1)^n n^p$ where $p = 1$.

Clearly the sequence is infinitely oscillating.

Ex. 2. Find the limits of the sequences whose nth terms are given below.

(*i*) $1 + \frac{1}{n^2}$ (*ii*) $3 + (-1)^n n^{-2}$.

Sol. (*i*) Let $a_n = 1 + \frac{1}{n^2}$.

Let (u_n) and (v_n) be two sequences such that $u_n = 1$, $v_n = \frac{1}{n^2}$ for all natural numbers.

Then $a_n = u_n + v_n$.

Also $\lim_{n\to\infty} u_n = 1$ and

$\lim_{n\to\infty} v_n = 0$.

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} u_n + \lim_{n\to\infty} v_n$

$= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \left(\frac{1}{n^2}\right)$

$$= 1 + 0 = 1.$$

$\therefore \lim_{n\to\infty} a_n = 1$

(*ii*) Put $a_n = 3 + (-1)^n n^{-2}$

Let $u_n = 3$ and $v_n = (-1)^n n^{-2}$

Then $\lim_{n\to\infty} u_n = 3$ and

$\lim_{n\to\infty} v_n = 0.$

Also $a_n = v_n + v_n$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} u_n + \lim_{n\to\infty} v_n$$

$$= \lim_{n\to\infty} 3 + \lim_{n\to\infty} (-1)^n n^{-2}$$

$$= 3 + 0 = 3.$$

Ex. 3. Obtain the limits of the sequences whose nth terms are given below :

(*i*) $\dfrac{n^2 + 3n}{2n^2 + 1}$; (*ii*) $\dfrac{n + 1}{n^2 + 2n}$ (*iii*) $\dfrac{1+2+3\cdot+\ldots+n}{n^2}$

Sol. (*i*) Let $a_n = \dfrac{n^2 + 3n}{2n^2 + 1}$.

Dividing the numerator and the demoninator of a_n by n^2, we get

$$a_n = \frac{1 + \frac{3}{n}}{2 + \frac{1}{n^2}} .$$

Now,

$$\lim_{n\to\infty} \left(2 + \frac{1}{n^2} \right) = \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n^2}$$

$$= 2 + 0 = 2 \neq 0$$

and $\left(2 + \dfrac{1}{n^2} \right) \neq 0$ for all n.

$$\therefore \lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} \left(1 + \frac{3}{n} \right)}{\lim_{n\to\infty} \left(2 + \frac{1}{n^2} \right)}$$

$$= \frac{1 + 0}{2 + 0} + \frac{1}{2} .$$

In this example, we have used Thm. 5.

(*ii*) Let $a_n = \dfrac{n + 1}{n^2 + 2n}$.

Dividing the numerator and the denominator of a_n by n^2, we get

$$a_n = \frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{2}{n}}$$

Also, $\lim_{n\to\infty} \left(1 + \frac{2}{n}\right) = 1 + 0 = 1 \neq 0$

and $1 + \frac{2}{n} \neq 0$ for all n.

$\therefore$ Applying Thm. 5 we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{2}{n}}\right)$$

$$= \frac{\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n^2}\right)}{\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)}$$

$$= \frac{0 + 0}{1 + 0} = 0$$

$\therefore \lim_{n\to\infty} a_n = 0.$

(*iii*) Put $a_n = \frac{1 + 2 + 3 \ldots + n}{n^2}$

$$= \frac{\left[\frac{n(n+1)}{2}\right]}{n^2} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}.$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right)$$

$$= \tfrac{1}{2} + 0 = \tfrac{1}{2}.$$

Note : Whenever a_n is the ratio of two polynomials in n, to obtain the limit of the sequence quickly, it is advisable to divide the numerator and the denominator of a_n by n^p where p is highest index (power) of n in the denominator. Afterwards, on taking the limits, one can easily obtain the limits of the sequence whenever it exists. It is observed that if the highest power of n in the denominator is greater than or equal to the highest power of n in the numerator, then the sequence is convergent.

Ex. 4. Find the limits of the sequences whose nth terms are given below :

(*i*) $\frac{2^n+3^{n-1}}{5^{n+2}}$; (*ii*) $\frac{3^n+2^{n+1}}{3^n+1}$; (*iii*) $\frac{5^{n+1}+2^n}{5^{n-1}+3^{n+1}}$.

Sol. (*i*) Put $a_n = \frac{2^n+3^{n-1}}{5^{n+2}}$.

Then $$a_n = \left(\frac{2^n}{5^n}\right)\frac{1}{5^2} + \left(\frac{3^n}{5^n}\right)\frac{3^{-1}}{5^2}$$

$$= \left(\frac{2}{5}\right)^n \frac{1}{25} + \left(\frac{3}{5}\right)^n \frac{1}{125}.$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\left(\frac{2}{5}\right)^n \times \frac{1}{25} + \left(\frac{3}{5}\right)^n \times \frac{1}{125}\right]$$

$$= \lim_{n\to\infty}\left(\frac{2}{5}\right)^n \times \frac{1}{25} + \lim_{n\to\infty}\left(\frac{3}{5}\right)^n \times \frac{1}{125}$$

$$= 0 \times \frac{1}{25} + 0 \times \frac{1}{125}$$

$$= 0 + 0 = 0.$$

$\therefore$ The sequence (a_n) converges to 0.

(*ii*) Let $a_n = \frac{3^n+2^{n+1}}{3^n+1}$

Divide the numerator and the denominator of a_n by 3^n. Then, we get

$$a_n = \frac{1+\frac{2^n}{3^n}\times 2}{1+\frac{1}{3^n}} = \frac{1+\left(\frac{2}{3}\right)^n \times 2}{1+\left(\frac{1}{3}\right)^n}$$

Now, $1+(\frac{1}{3})^n \neq 0$ for all n and also $\lim_{n\to\infty}[1+(\frac{1}{3})^n] = 1+0 = 1$

$$\therefore \lim_{n\to\infty} a^n = \lim_{n\to\infty}\left[\frac{1+(\frac{2}{3})^n\times 2}{1+(\frac{1}{3})^n}\right]$$

$$= \frac{\lim_{n\to\infty}[1+(\frac{2}{3})^n\times 2]}{\lim_{n\to\infty}[1+(\frac{1}{3})^n]}$$

$$= \frac{1+0}{1+0} = \frac{1}{1} = 1.$$

$\therefore$ The sequence (a_n) converges to 1.

(*iii*) Let $a_n = \frac{5^{n+1}+2^n}{5^{n-1}+3^{n+1}}$

Divide the numerator and the denominator of a_n by 5^n. Then, we get

$$a_n = \frac{5 + \frac{2^n}{5^n}}{\frac{1}{5} + \frac{3^n}{5^n} \times 3} = \frac{5 + \left(\frac{2}{3}\right)^n}{\frac{1}{5} + \left(\frac{3}{5}\right)^n \times 3}$$

Now, $\frac{1}{5} + (\frac{3}{5})^n \times 3 \neq 0$ for all n and $\lim [\frac{1}{5} + (\frac{3}{5})^n \times 3] = \frac{1}{5} + 0 = \frac{1}{5}$.

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[\frac{5 + (\frac{2}{3})^n}{\frac{1}{5} + (\frac{3}{5})^n \times 3}\right]$$

$$= \frac{\lim_{n\to\infty} [5 + (\frac{2}{3})^n]}{\lim_{n\to\infty} [\frac{1}{5} + (\frac{3}{5})^n \times 3]}$$

$$= \frac{5 + 0}{\frac{1}{5} + 3 \times 0} = \frac{5}{(\frac{1}{5})} = 25.$$

$\therefore$ The sequence converges to 25.

Note: In case of Ex. 4, it can be seen that the numerator and denominator of each term of the sequence consists of the sum of the terms of the type a^{n+p} where p is some constant. In case of the number a^{n+p}, we call the number a as the *base* and $(n + p)$ as the index. If the terms of the sequence are of the type given in Ex. 4, it is advisable to divide the numerator and the denominator by x^n where x is the greatest base number among the terms appearing in the denominator. This helps to evaluate the limit of the sequence quickly.

Ex. 5. Evaluate the limits of the sequences whose nth terms are given below.

(*i*) $\sqrt{n^2+1} - \sqrt{n^2-1}$; (*ii*) $\log_2\left(\frac{2n^2+2n+5}{2+5n+n^2}\right)$;

(*iii*) Sin $(2^{1/n})$.

Sol. (*i*) Let $a_n = \sqrt{n^2+1} - \sqrt{n^2-1}$.

We are aware that

$\lim_{\to\infty} \sqrt{n^2+1} = +\infty$ and

$\lim_{n\to\infty} \sqrt{n^2-1} = +\infty$

Therefore, apparently it seems that $\lim_{n\to\infty} a_n$ does not exist. However, if we rationalize the term, then we observe that the limit does exist.

$a_n = \sqrt{n^2+1} - \sqrt{n^2-1}$

$$= \left(\sqrt{n^2+1} - \sqrt{n^2-1}\right) \times \frac{\left(\sqrt{n^2+1} + \sqrt{n^2-1}\right)}{\left(\sqrt{n^2+1} + \sqrt{n^2-1}\right)}$$

$$= \frac{(n^2+1) - (n^2-1)}{\sqrt{n^2+1} + \sqrt{n^2-1}}$$

$$= \frac{2}{\sqrt{n^2+1} + \sqrt{n^2-1}}$$

Now dividing the numerator and the denominator by $n = \sqrt{n^2}$, we get

$$a_n = \frac{(2/n)}{\sqrt{1+\frac{1}{n^2}} + \sqrt{1-\frac{1}{n^2}}}$$

Also $\lim_{n\to\infty} \left(\sqrt{1+1/n^2} + \sqrt{1-1/n^2}\right) = \sqrt{1} + \sqrt{1} = 2$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[\frac{(2/n)}{\sqrt{1+1/n^2} + \sqrt{1-1/n^2}}\right]$$

$$= \frac{0}{2} = 0.$$

$\therefore$ The sequence (a_n) converges to 0.

(*ii*) Let $a_n = \log_2 \left(\frac{2n^2+2n+5}{2+5n+n^2}\right)$

Now, we know that

$$\lim_{n\to\infty} \left(\frac{2n^2+2n+5}{2+5n+n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2+\frac{2}{n}+\frac{5}{n^2}}{\frac{2}{n^2}+\frac{5}{n}+1}\right) = \frac{2+0+0}{0+0+1} = \frac{2}{1} = 2.$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left\{\log_2 \left(\frac{2n^2+2n+5}{2+5n+n^2}\right)\right\}$$

$$= \log \left\{\lim \left[\frac{2n^2+2n+5}{2+5n+n^2}\right]\right\}$$

..... by Thm. 6 (*iv*)

$$= \log_2 2 = 1.$$

$$\therefore \lim_{n\to\infty} a_n = 1.$$

$\therefore$ The sequence (a_n) is convergent to 1.

(*iii*) Let $a_n = \text{Sin}\,(2^{1/n})$.

Now, we are aware that

$$\lim_{n\to\infty}(2^{1/n}) = 1.$$

$$\therefore \quad \lim_{n\to\infty} \text{Sin}\,(2^{1/n}) = \text{Sin}\,(\lim_{n\to\infty} 2^{1/n})$$

$$= \text{Sin}\ 1.$$

$\therefore$ The sequence (a_n) is convergent to Sin 1.

EXERCISE 1·2.

1. Examine whether the sequences, whose nth terms are given below, are convergent, divergent or oscillating.

(*i*) $\frac{1}{n}$; (*ii*) n^3 ; (*iii*) $3+\frac{(-1)^n}{n^4}$; (*iv*) $(-1)^n\times 2$; (*v*) $(-1)^n 2^n$.

(*vi*) $1+(-1)^{n+1}$; (*vii*) $n+(-1)^n$; (*viii*) $1+\frac{1}{n^2}$; (*ix*) $3-\frac{1}{n^3}$;

(*x*) $4+\frac{1}{n}$; (*xi*) $2-n^2$; (*xii*) $3-4n$; (*xiii*) $2^{1/n}$; (*xiv*) $(3\,5)^{1/n}$

2. Prove that the sequences, whose nth terms are given below, are convergent and evaluate their limits.

(*i*) $\frac{n+2}{n}$; (*ii*) $\frac{2n+3}{3n}$; (*iii*) $\frac{n+2}{4n+1}$; (*iv*) $\frac{3n-2}{2-5n}$; (*v*) $\frac{1-n}{2n+1}$;

(*vi*) $\frac{n^2+3}{2n^2+1}$; (*vii*) $\frac{n^2-n+1}{4-2n+n^2}$; (*viii*) $\frac{n^2+5n+2}{4n^2+3n+1}$; (*ix*) $\frac{n^3+1}{3n^3+2}$

(*x*) $\frac{n^3-3n^2+1}{4-2n-n^3}$; (*xi*) $\frac{7n^3+2n^2+5n+2}{4n^3-n^2-4n+5}$; (*xi*) $\frac{1^2+2^2+3^2+\ldots+n^2}{(n+1)(n+2)(n+3)}$;

(*xiii*) $\frac{n^2+5n+6}{1+2+3+\ldots+n}$.

3. Prove that the sequences, whose nth terms are given below, converge to zero.

(*i*) $\frac{n-1}{n^2}$; (*ii*) $\frac{2n+3}{n^2+3}$; (*iii*) $\frac{n^2+5n-1}{1+n-3n^2}$; (*iv*) $\frac{1-n^2}{1+n^3}$; (*v*) $\frac{3n-2}{3n^3+2n}$

4. Show that the sequences, whose nth terms are given below, are divergent. Point out which sequences diverge to $+\infty$ and which diverge to $-\infty$.

(*i*) $\frac{n^2+1}{n}$; (*ii*) $\frac{n^2+3n+5}{2n+1}$; (*iii*) $\frac{3n+2n^2-n^3}{2n^2+3n+2}$; (*iv*) $\frac{n^2-7n+4}{4n+3}$;

(*v*) $\frac{3n-n^4}{1+n+n^2}$; (*vi*) $\frac{n+1}{\sqrt{n}+\sqrt[3]{n}}$; (*vii*) $\frac{n^2-5n^3+n}{n\sqrt{n}+n^2}$;

(*viii*) $\frac{\sqrt{n}+\sqrt[3]{n}}{\sqrt[4]{n}+\sqrt[5]{n}}$

5. Find the limits of the sequences whose nth terms are given below.

(*i*) $\frac{1+2+3+\ldots+n}{n^2+1}$; (*ii*) $\frac{1+3+5+\ldots+(2n-1)}{1+2+3+\ldots+n}$;

(*iii*) $\frac{1^3+2^3+3^3+\ldots+n^3}{n^2(2n^2+1)}$; (*iv*) $\frac{1^2+2^2+\cdots+n^2}{n^3}$;

(*v*) $\frac{1+4+7+\ldots+(3n-2)}{1+3+5+\ldots+(2n-1)}$; (*vi*) $\frac{3+7+11+\ldots+(4n-1)}{1+4+7+\ldots+(3n-2)}$;

6. Examine the sequences, whose nth terms are given below, for convergence and obtain their limits whenever they exist.

(*i*) $\sqrt{n+2}-\sqrt{n}$; (*ii*) $\sqrt{n^2+n+1}-\sqrt{n^2+1}$

(*iii*) $\sqrt{n+4}-\sqrt{n-2}$; (*iv*) $\sqrt{n^2+2n+3}-\sqrt{n^2+n+1}$;

(*v*) $(\sqrt{n+3}-\sqrt{n-1})\sqrt{n}$;

(*vi*) $(\sqrt{n-2}-\sqrt{n-5})\sqrt{n}$;

(*vii*) $\sqrt{2n^2+3n+2}-\sqrt{2n^2-n-3}$.

7. Find the limits of the sequences, whose nth terms are given below, whenever they exist.

(*i*) $(0{\cdot}1)^n$; (*ii*) 3^{-n}; (*iii*) $\frac{2^n+3^n}{1+2^n}$; (*iv*) $\frac{3^{n+1}-2^n}{3^{n+1}+2^n}$;

(*v*) $\frac{1+3\,.\,2^n}{1-2^{n+1}}$; (*vi*) $\frac{2^{2n}}{3^{3n}}$; (*vii*) $\frac{5^{n+1}+6^{n+2}}{5^{n-1}-6^{n-2}}$; (*viii*) $\left(\frac{3}{2}\right)^n$

(*ix*) $1+2^{-n}$ (*x*) $\left(-\frac{1}{3}\right)^n$; (*xi*) $1+3^{-n}-2^{-n}$; (*xii*) $1+\frac{(-1)^n}{2n}$;

(*xiii*) $2-\left(\frac{-1}{5}\right)^n$; (*xiv*) $\sqrt[n]{5}$; (*xv*) $\sqrt[n]{2+\sqrt{3}}$.

(*xvi*) $\frac{5n^{-2}+3n^{-3}}{7n^{-2}+2n^{-3}}$; (*xvii*) $\frac{2^{n+1}+3^{n-3}}{4^{n-1}+5^{n+2}}$;

(*xviii*) $\frac{5\times 2^{n-3}-8\times 3^{n+2}+6\times 7^{n+1}}{4\times 5^n+27\times 6^{n-1}+9\times 7^{n-2}}$

8. Find the limits of the sequences whose nth terms are given below.

(*i*) $2^{\left(\frac{n+1}{2n}\right)}$ (*ii*) $4^{\left(\frac{2n+3}{5n^2}\right)}$ (*iii*) $9^{\left(\frac{2n+5}{4n+9}\right)}$

(*iv*) $\log_4\left(\frac{2n+5}{n+3}\right)$; (*v*) $\log_{10}\left(\frac{n^2+5n+6}{n^2-6n+9}\right)$;

(*vi*) $(-1)^n \tan\left((-1)^n\frac{\pi}{4}\right)$; (*vii*) $\sin\left(\frac{n^2+2n+5}{n^3+5n+9}\right)$;

(*viii*) $\cos\left(\frac{n^2+6n+7}{3n^3+6n^2+11}\right)$; (*ix*) $\mathrm{Cos}\left[\mathrm{Sin}\left(\frac{n^2+2n+5}{n^3+5n+9}\right)\right]$;

ANSWERS

1. (*i*) 0 ; (*ii*) $+\infty$; (*iii*) 3 ; (*iv*) Finitely oscillating ; (*v*) Infinitely oscillating ; (*vi*) Finitely oscillating ; (*vii*) diverges to $+\infty$; (*viii*) 1 ; (*ix*) 3 ; (*x*) 4 ; (*xi*) diverges to $-\infty$; (*xiii*) diverges to $-\infty$; (*xiv*) 1 ; (*xv*) 1.

2. (*i*) 1 ; (*ii*) $\frac{2}{3}$; (*iii*) $\frac{1}{4}$; (*iv*) $-\frac{3}{5}$; (*v*) $-\frac{1}{2}$; (*vi*) $\frac{1}{2}$;

(*vii*) -1 ; (*viii*) $-\frac{1}{4}$; (*ix*) $\frac{1}{3}$; (*x*) -1 ; (*xi*) $\frac{7}{4}$; (*xii*) $\frac{1}{3}$;

(*xiii*) 2.

4. (*i*) $+\infty$; (*ii*) $+\infty$; (*iii*) $-\infty$; (*iv*) $+\infty$; (*v*) $-\infty$; (*vi*) $+\infty$; (*vii*) $-\infty$; (*viii*) $+\infty$;

5. (*i*) $\frac{1}{2}$; (*ii*) 2 ; (*iii*) $\frac{1}{8}$; (*iv*) $\frac{1}{3}$; (*v*) $\frac{3}{2}$; (*vi*) $\frac{4}{3}$.

6. (*i*) 0 ; (*ii*) $\frac{1}{2}$; (*iii*) 0 ; (*iv*) $\frac{1}{2}$; (*v*) 2 ; (*vi*) $\frac{3}{2}$; (*vii*) $\sqrt{2}$

7. (*i*) 0 ; (*ii*) 0 ; (*iii*) diverges to $+\infty$; (*iv*) $\frac{1}{9}$; (*v*) $-\frac{3}{2}$; (*vi*) 0 ; (*vii*) 6^4 ; (*viii*) diverges to $+\infty$; (*ix*) 0 ; (*x*) 0 ; (*xi*) 1 ; (*xii*) 1 ; (*xiii*) 2 ; (*xiv*) 1 ; (*xv*) 1 ; (*xvi*) $\frac{5}{7}$; (*xvii*) 0 ; (*xviii*) $\frac{2}{21}$

8. (*i*) $\sqrt{2}$; (*ii*) 1 ; (*iii*) 3 ; (*iv*) $\frac{1}{2}$; (*v*) 0 ; (*vi*) 1 ; (*vii*) 0 ; (*viii*) 1 ; (*ix*) 1 ; (*x*) 1.

1·8. Bounded and Unbounded Sets.

Consider the sets A, B, C, D given as

$A=\{3, 4, 5, 6, 7, 8, 9, 10\}$,

$B=\{2, 3, 4, 5, 6, \ldots\ldots\}=\{n+1 \,/\, n \in \mathbb{N}\}$,

$C=\{\ldots\ldots, -3, -2, -1\}$=The set of all negative integers, and

$D=\{-3, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$,

All these four sets are sets of real numbers. These sets have some interesting properties as mentioned below.

(*i*) No element of A is larger than 10. Also if α is any real number larger than 10, then it is clear that *no element* of A is larger than α. Similarly no element of the set A is smaller than 3. Also if β is any real number smaller than 3, then it is evident that no element of A is smaller than β.

(*ii*) In case of set B, no element of B is less than 2, Also if β is any number smaller than 2, then *no element* of B is found to be smaller than β also. However, we cannot find a number α such that no element of B is larger than α.

(*iii*) In case of set C, it is observed that no element of C is larger than -1. Also, it can be verified that if α is any number larger than -1, then no element of C is found to be larger than α. However we cannot obtain a real number β such that no element of C is less than β.

(*iv*) In case of Set D, we can not find real numbers α and β such that no element of D is greater than α and no element of D is less than β.

Then sets of the type B are said to be *bounded below*, the sets of the type C are said to be *bounded above*, the sets of the type A

are said to *bounded* (bounded above and bounded below) and sets of the type D are said to be *unbounded*. We shall now give formal definitions of these concepts.

Def. 1. 8. 1. A nonempty set X of real numbers is said to be *bounded above* if there exists a real number α such that

$x \leqslant \alpha$ for all x in X.

Def. 2. *A* nonempty Set X of real numbers is said to be *bounded below* if there exists a real number β such that

$\beta \leqslant x$ for all x in X.

Def. 3. *A* nonempty set X is said to be *bounded* if it is bounded above and bounded below.

Def. 4. *A* nonempty set X is said to be *unbounded* if it is neither bounded above nor bounded below.

It is observed that if X is any nonempty set of real number, then only *one* of the following statements is valid for X.

(*i*) X is bounded,

(*ii*) X is bounded above but not bounded below,

(*iii*) X is bounded below but not bounded above, and

(*iv*) X *is unbounded.*

The following examples will further clarify these concepts.

Ex. 1. Classify the boundedness of the following sets of real numbers.

(*i*) $A=\{x \mid -2<x<5\}$

(*ii*) $B=\{x \mid 2 \leqslant x\}$

(*iii*) $C=\left\{\frac{1}{n} \middle/ n \in \mathbb{N}\right\}$

(*iv*) $D=\{x \mid x \leqslant 0\}$

(*v*) $E=\{x \mid x \in \mathbb{R}\}$

(*vi*) $F=\{x \mid x<3\}$

(*vii*) $G=\{x \mid 2 \leqslant x \leqslant 5\}$

Sol. (*i*) A is a bounded set.

(*ii*) B is bounded below but not bounded above.

(*iii*) C is a bounded set.

(*iv*) D is bounded above but not bounded below.

(*v*) E is an unbounded set. *i.e.* E is neither bounded above nor bounded below.

(*vi*) F is bounded above but not bounded below.

(*vii*) G is a bounded set.

Least upper bounds and greatest lower bounds

Let A be any nonempty set of real numbers.

Def. A real number p is called an upper bound of A if $x \leqslant p$ for all x in A.

Def. A real number q is called a lower bound of A if $q \leqslant x$ for all x in A.

Consider the set A given by

$A = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$

Clearly, the set A is bounded above and bounded below. We may take -4 to be its *lower* bound and 7 as its upper bound as any number x of A satisfies the relation.

$-4 < x < 7$.

However, we observe that any number greater than 7, say 7·1 or 8 or 100 or 1000 or 59200 etc is also an upper bound of A. Similarly, it can be verified that any number less than -4 is also a lower bound of A. This means, if A has an upper bound then A has infinitely many upper bounds ; and if A has a lower bound then A has infinitely many lower bounds.

We may summarize these observations as the following remark.

Remark. (*a*) If a nonempty set A of real numbers is bounded above, then A has *at least one* upper bound, and consequently A has infinitely many upper bounds.

(*b*) If a nonempty set A of real numbers is bounded below, then A has *at least one* lower bound, and this implies that A has infinitely many lower bounds.

Now, consider the interval I,

$I = (2, 3) = \{x / 2 < x < 3,\ x \in /R\}$

For this set, 2, 1·5, 1·2, -3, 0, etc are lower bounds, But we can show that 2 is the greatest, among all, these lower bounds. Since $2 \leqslant x$ for all x in I, 2, qualifies to be a lower bound.

Now, if $y > 2$ is a lower bound of I, then $y \leqslant x$ for all x in I. However, since $2 < y$, $2 < \frac{y+2}{2} < y$ and hence $z = \frac{y+2}{2}$ is an element of I. This means that I contains an element $z = \frac{y+2}{2}$ which is greater

than 2 and further $z<y$. Hence, certainly y can not be a lower bound of I. This anamoly has resulted because we made a wrong assumption that $y>2$ can be a lower bound of 2. This means that no number larger than 2 can be a lower bound of I. Thus, we observe that 2 is a lower bound of I and 2 is the greatest among all lower bounds of I. *i.e.* 2 is the *greatest lower bound of I.*

In the same way, we can verify that 3 is an upper bound of I and 3 is the least (smallest) among all upper bounds of I. *i.e.* 3 is the *least upper bound* of I.

We shall now define mathematically the greatest lower bound and the least upper bound of a nonempty set.

Def. 5. Least Upper Bound (lub)

A real number p is said to be the least upper bound of nonempty set A of real numbers if

(*i*) p is an upper bound of A and

(*ii*) no upper bound of A is less than p.

Def. 6. Greatest Lower Bound (glb)

A real number q is said to be the greatest lower bound of a nonempty set H of real numbers if

(*i*) q is a lower bound of A and

(*ii*) no lower bound of A is greater than q.

Ex. 2. Obtain the glb and lub of the following sets whenever they exist.

(*i*) $A=\{-3, -2, -1, 0, 1, 2, 3, 4, 5\}$

(*ii*) $B=[2, 3)=\{x/2\leqslant x<3;\ x \in \mathbb{R}\}$

(*iii*) $C=(-3, 7)=\{x/ -3<x\leqslant 7\ ;\ x \in \mathbb{R}\}$

(*iv*) $D=[0, \infty)=\{x/0\leqslant x\ ;\ x \in \mathbb{R}\}$ and

(*v*) $E=(-\infty, -5]=\{x/x\leqslant -5\ ;\ x \in \mathbb{R}\}$ and

(*vi*) $F=$Set of all real numbers $=(-\infty, \infty)$.

Sol. (*i*) The glb and lub of A are respectively -3 and 5.

(*ii*) The glb and lub of B are respectively 2 and 3.

(*iii*) The glb and lub of C are respectively -3 and 7.

(*iv*) The glb of D is 0. However, the set D has no lub.

(*v*) The lub of E is -5. But the set E has no glb.

(*vi*) The set F has neither lub nor glb.

Now, we observe that some sets have lub but no glb ; some sets have glb but no lub, some sets have both lub and glb while some sets have neither lub nor glb. We shall therefore like to know

the conditions under which a given and nonempty set of real numbers has the lub and glb.

We give below the criteria under which a nonempty set of real numbers has the lub. and glb in the form of *axiom.*

Axiom of least upper bound

Every nonempty set of real numbers bounded above has the *least upper bound.*

It can be shown the same is equivalent to the following.

Every nonempty set of real numbers bounded below has the *greatest lower bound.*

It is worth noting that the lub of a set bounded above may or may not be an element of the set. In the same way, the glb of a set bounded below may or may not be an element of the set. The following examples illustrates this aspect completely.

The open interval $(-2, 6)$ is both bounded above and below. Therefore, by above axioms, it has the lub and glb. It is an easy exercise to check that -2 is the glb and 6 is the lub of this open interval $(-2, 6)$. But neither -2 nor 6 is an element of $(-2, 6)$.

The set $E=\{-3, -2, -1, 0, 1, 2, 3, 4, 5\}$ is both bounded above and below and hence it has the lub and glb. It is observed that -3 is the glb and 5 is the lub of E. In this case both -3 and 5 are the elements of the set.

The set $F=\{x \mid -2<x \leqslant 11,\ x \in \mathbb{R}\}$ is both bounded above and below and therefore it has the lub and glb. It can be verified that -2 is the glb and 11 is the lub of F. However, the glb -2 is not an element of F though the lub 11 is an element of F.

Ex. 3. Find the glb and lub of the following sets whenever they exist. Examine whether any of them, whenever they exist, belongs to the set.

(*i*) $A=\{x/x=n^2\ ;\ n \in \mathbb{N}\}$

(*ii*) $B=\{x/x=1+\frac{1}{n}\ ;\ n \in \mathbb{N}\}$

(*iii*) $C=\{x/x=2+\frac{(-1)^n}{n}\ ;\ n \in \mathbb{N}\}$

(*iv*) $D=(x/x=(-1)^n\ 2\ ;\ n \in \mathbb{N}\}$

(*v*) $E=\{x/x=(-1)^n\ n\ \in \mathbb{N}\}$

Sol. (*i*) The set A can be written as $A=\{1, 4, 9, 16, 25, 36, 49, \ldots, n^2, \ldots\}$. Clearly the set A is bounded below but not bounded

above. The glb of the set A is 1 and it is an elment of A. A has no lub.

(*ii*) The Set B can expressed as $B=\left\{1+\frac{1}{1}, 1+\frac{1}{2}, 1+\frac{1}{3}\right.$ $\left.1+\frac{1}{4}, .., 1+\frac{1}{n}...,\right\}$. It shows that B is bounded above and below. The lub of B is 2 and it is an element of B. The glb of B is 1 but it is not an element of B.

(*iii*) We can write the set C as $C=\left\{2-\frac{1}{1}, 2+\frac{1}{2}, 2-\frac{1}{3},\right.$ $\left.2+\frac{1}{4}, ...\right\}$. which on rearrangement, can be put as $C=\left\{2-\frac{1}{1}\right.$ $\left.2-\frac{1}{3}, 2-\frac{1}{5}, ...,2+\frac{1}{6}, 2+\frac{1}{4}, 2+\frac{1}{2}\right\}$. This reveals that C is a bounded set and the lub of C is $2+\frac{1}{2}=\frac{5}{2}$ and the glb of Set C is $2-\frac{1}{1}=1$. Both the lub and glb are the elements of C.

(*iv*) $D=\{(-1)^n\ 2\ /n \in \mathbb{N}\}=\{-2, 2\}$. The glb of D is -2 and the lub of D is 2 and both lub and glb of D are the elements of D.

(*v*) We can express the Set E as $E=\{....., ..., -7, -5, -3, -1, 2, 4, 6, 8, 10, ...\}$. Since E is neither bounded above nor bounded below, E has no lub and no glb.

Ex. 4. Examine the following sets for boundedness. Find their lub and glb whenever they exist and determine whether they belong to the set.

(*i*) $A=\{x/x^2-8x+15<0\ ;\ x \in \mathbb{R}\}$

(*ii*) $B=\{x/x^2-3x+2\geqslant 0\ ;\ x \in \mathbb{R}\}$

Sol. (*i*) $A=\{x/x^2-8x+15<0\ ;\ x \in \mathbb{R}\}$

We have, $x^2-8x+15=(x-3)\ (x-5)$

$\therefore$ $x^2-8x+15<0$ whenever $(x-3)$ and $(x-5)$ have opposite signs.

Now, if $x-3\leqslant 0$, then $x-5\leqslant 0$.

Also, if $x-5\leqslant 0$, Then $x-3\geqslant 0$.

Therefore, whenever $3<x<5$ then $(x-3)>0$ and $(x-5)<0$. Consequently $(x-3)$ and $(x-5)$ have opposite signs only when $3<x<5$.

$\therefore$ $x^2-8x+15<0$ only when $3<x<5$

$\therefore$ $A=\{x/x^2-8x+15<0\ ;\ x \in \mathbb{R}\}$

$=\{x/3<x<5\ x\ \in \mathbb{R}\}$

$=(3, 5)$.

Clearly A is bounded, the lub of A to 5, the glb of A is 3. Both lub and glb *do not* belong to the set A.

(ii) $B=\{x\ /x \in /\mathbb{N}\ ;\ x^2-3x+2\geqslant 0\}$.

We have, $x^2-3x+2=(x-1)\ (x-2)$.

$\because\ x^2-3x+2\geqslant 0$ whenever $(x-1)$ and $(x-2)$ have the same signs or when either of them is zero.

Solving $x^2-3x+2=0$, we observe that $x=1$ or $x=2$.

Also, when $x<1$ then $(x-1)<0$ and $(x-2)<0$ and hence $(x-1)$ and $(x-2)$ have the same signs. Similarly, when $x>2$, then $(x-1)\ >0$ and $(x-2)>0$ and hence they have the same signs.

But when $1<x<2$, then $(x-1)>0$ and $(x-2)<0$, *i.e.* $(x-1)$ and $(x-2)$ have opposite signs.

Consolidating above arguments, we observe that

$B=\{x/x \in \mathbb{R}\ ;\ x\leqslant 1$ or $x\geqslant 2\}$

$\therefore$ B is not a bounded set. Therefore B has neither lub nor glb.

✱ EXERCISE 1·3.

1. Examine the following sets for boundedness. State the *glb* and *lub* when they exist and find whether they belong to the corresponding sets.

(i) $A = \{x/x \in \mathbb{R}\ ;\ x\geqslant -2\}$

(ii) $B = \{x/x \in \mathbb{R}\ ;\ x>0\}$;

(iii) $C = \{x/x \in \mathbb{R}\ ;\ x<10\}$;

(iv) $D = \{x/x \in \mathbb{R}\ ;\ x\leqslant -2\}$;

(v) $E = \{x/x \in \mathbb{R}\ ;\ -1<x\leqslant 2\}$;

(vi) $F = \{x/x \in \mathbb{R}\ ;\ 2<x<5\}$;

(vii) $G = \{x/x \in \mathbb{R}\ ;\ 0\leqslant x<7\}$ and

(viii) $H = \{x/x \in \mathbb{R}\ ;\ -3\leqslant x<3\}$.

2. Write down the following sets by the roster method. Hence or otherwise find out whether they are bounded. State the *g.l.b.* and the *lub* when they exist and find out whether they are elements of the corresponding sets.

(i) $A = \{x/x = 1-2n,\ n \in \mathbb{N}\}$.

(ii) $B = \{x/x = (-1)^n\ 2^n,\ n \subset \mathbb{N}\}$

(iii) $C = \{x/x = 2n+1\ ;\ n \in \mathbb{N}\}$;

(iv) $D = \{x/x = (-1)^n,\ n \in \mathbb{N}\}$;

(v) $E = \{x/x = \sin \frac{n\pi}{6},\ n \in \mathbb{N}\}$;

(vi) $F = \{x/x = (\frac{1}{2})^n\ ;\ n \in \mathbb{N}\}$;

(vii) $G = \{x/x = \frac{3n+1}{n}\ ;\ n \in \mathbb{N}\}$;

(*viii*) $H = \{x/x = 3 - \frac{(-1)^n}{n} ; n \in \mathbb{N}\}$;

(*ix*) $I = \{x/x = 2 - \frac{1}{n} ; n \in \mathbb{N}\}$ and

(*x*) $J = \{x/x = \frac{(-1)^n}{3n} ; n \in \mathbb{N}\}$.

3. Examine the following sets for boundedness. Find their *g. l. b.* and *l. u. b.* and determine whether they belong to the corresponding sets.

(*i*) $A = \{x/x \in \mathbb{R} ; (x-2)(x-5) \leqslant 0\}$;

(*ii*) $B = \{x/x \in \mathbb{R}; x(x-3) > 0\}$;

(*iii*) $C = \{x/x \in \mathbb{R} ; (x+2)(x-4) < 0\}$;

(*iv*) $D = \{x/x \in \mathbb{R} ; x^2 - 5x + 6 \leqslant 0\}$;

(*v*) $E = \{x/x \in \mathbb{R} ; x^2 - 4x > 0\}$ and

(*vi*) $F = \{x/x \in \mathbb{R} ; 2x^2 - 5x + 3 < 0\}$.

ANSWERS

Key : *m* stands for the glb of the set
M stands for the lub of the set.

1. (*i*) $m = -2 \in A$; (*ii*) $m = 0 \notin B$. (*iii*) $M = -10 \in C$
(*iv*) $M = -2 \in D$. (*v*) $m = -1 \notin E$, $M = 2 \in E$.
(*vi*) $m = 2 \notin F$, $M = 5 \notin F$. (*vii*) $m = 0 \in G$; $M = 7 \notin G$;
(*viii*) $m = -3 \in H$, $M = 3 \in H$;

2. (*i*) $M = -1 \in A$; (*ii*) Unbounded set. (*iii*) $m = 3 \in C$,
(*iv*) $m = -1$, $M = 1$ both $\in D$; (*v*) $m = -1$. $M = 1$ both belong to E (*vi*) $m = 0 \notin F$, $M = \frac{1}{2} \in F$. (*vii*) $m = 3 \notin G$ $M = 4 \in G$,(*viii*) $m = 2$, $M = 3{\cdot}5$; both belong to H. (*ix*) $m = 1 \in I$, $M = 2 \notin I$, (*x*) $m = -\frac{1}{3}$, $M = \frac{1}{9}$ both belong to J.

3. (*i*) $m = 2$, $M = 5$; both belong to A. (*ii*) unbounded set,
(*iii*) $m = -2$, $M = 4$; both are not in C ; (*iv*) $m = 2$, $M = 3$;both belong to D. (*v*) unbounded set ; (*vi*) $m = 1$, $M = \frac{3}{2}$; both belnog to E.

1·9. Bounded and Unbounded Sequences

With every sequence, we associate a set consisting of the terms of the sequence. Thus, with the sequence (a_n), we associate the set

$$\{x/x = a_n ; n \in \mathbb{N}\}.$$

The sequence (a_n) is said to be *bounded* if the set of terms of the sequence is bounded. If the set of terms of the sequence has the lub, then it is called the *lub of the sequence*. Similarly, if the set of terms of the sequence has the glb, then it is called the *glb of the sequence*.

Illustrative Examples

Ex. 1. Let the sequence (a_n) be given by

$$a_n = n^2+1.$$

Then, the set of terms of the sequence (a_n) is

$$\{ 2, 5, 10, 17, 26, 37, 50,..... \}$$

Clearly the set is bounded below but not bounded above. The glb of the sequence is 2 and the sequence has no lub.

Ex. 2. For the sequence (a_n) ; $a_n = 1-2n$, the set of the terms of the sequence is

$$\{-1, -3, -5, -7, -9,.....\}.$$

It is evident that the set is bounded above but not bounded below. Therefore, the lub of the sequence is -1 but the sequence has no glb.

Ex. 3. If the sequence (a_n) is defined by

$a_n = (-1)^n n$, then the set of terms of the sequence (a_n) is

$$\{-1, 2, -3, 4, -5, 6, -7, 8, -9, 10, ...\}$$

which on rearrangement can be written as

$$\{....., -7, -5. -3, -1, 2, 4, 6, 8, 10, ...\}$$

Clearly the set is not bounded. Therefore the sequence (a_n) is not bounded.

Ex. 4. Let the sequence (a_n) be defined as

$a_n = 1 + \dfrac{(-1)^n}{n}$. Then the set of the terms of the sequence is

$$\{0, 1+\tfrac{1}{2}, 1-\tfrac{1}{3}, 1+\tfrac{1}{4}, 1-\tfrac{1}{5}, 1+\tfrac{1}{6}, ...\}$$

Clearly, the set is bounded and its glb and lub are respectively 0 and $\frac{3}{2}$.

Therefore, the sequence is bounded and its glb and lub are respectively 0 and $\frac{3}{2}$.

Ex. 5 For the sequence (a_n) given by $a_n = (-1)^n 3$, the set of the terms of the sequence is

$$\{-3, 3, -3, 3, -3, 3, ...\} = \{-3, 3\}.$$

Clearly the set is bounded. Therefore the sequence is bounded and its glb and lub are respectively -3 and 3.

Remark. (*i*) In Ex. 1, 2 and 3, the sequences are not bounded above or below or are unbounded. However *none* of these sequences

is convergent *i.e.*, either they are divergent or they oscillate infinitely.

(*ii*) In Ex. 4, 5 the sequences are bounded. However, in Ex. 4, the sequence is convergent whereas in Ex. 5, the sequence is finitely oscillating.

Note : (*i*) A divergent sequence is either not bounded above or not bounded below.

(*ii*) An infinitely oscillating sequence is unbounded *i.e.*, it is neither bounded above *nor* bounded below.

(*iii*) A convergent sequence is a bounded sequence.

(*iv*) A bounded sequence is either convergent or finitely oscillating.

(*v*) If A sequence is either bounded above or bounded below, then it is either divergent or it is infinitely oscillating.

1·10. Monotonic Sequences

Def. 1. Monotone Increasing Sequence. A sequence (a_n) is said to be *monotone increasing sequence* if, and only if,

$$a_{n+1} - a_n \geqslant 0 \text{ for all } n.$$

Ex. 1. Let (a_n) be a sequence given by $a_n = 2$.

Clearly $a_{n+1} = a_n = 2$, for all n.

$\therefore \quad a_{n+1} - a_n = 2 - 2 = 0$. for all n.

$\therefore \quad a_{n+1} - a_n \geqslant 0$ for all n.

By above definition, (a_n) is a monotone increasing sequence.

Ex. 2. Let (a_n) be defined as $a_n = \dfrac{n}{n+1}$.

Then,

$$a_{n+1} - a_n = \frac{n+1}{(n+1)+1} - \frac{n}{n+1}$$

$$= \frac{(n+1)^2 - n\,[(n+1)+1]}{[(n+1)+1]\,(n+1)}$$

$$= \frac{1}{(n+1)\,(n+2)} > 0 .$$

$\therefore \quad a_{n+1} - a_n \geqslant 0$ for all n.

$\therefore$ (a_n) is a monotone increasing sequence.

Remark : If the sequence (a_n) is a monotonic increasing sequence of nonzero terms, then

$$a_{n+1} - a_n \geqslant 0 \text{ for all } n.$$

This implies that $a_{n+1} \geqslant a_n$ and consequently

$$\frac{a_{n+1}}{a_n} \geqslant 1 \text{ for all } n.$$

Sometimes this condition is used to define a monotone increasing sequence.

Def. 2. Monotone Decreasing Sequence. A sequence (a_n) is said to be *monotone decreasing* if, and only if,

$$a_{n+1} - a_n \leqslant 0 \text{ for all } n.$$

Ex. 3. Let (a_n) be defined as $a_n = 5$.

Clearly, $a_{n+1} = a_n = 5$ for all n.

$\therefore \quad a_{n+1} - a_n = 5 - 5 = 0.$

$\therefore \quad a_{n+1} - a_n \leqslant 0$ for all n.

$\therefore \quad (a_n)$ is a monotone decreasing sequence.

Ex. 4. The sequence (a_n) defined by $a_n = \frac{1}{n}$ is a monotone decreasing sequence because

$$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n-(n+1)}{n(n+1)}$$

$$= \frac{-1}{n(n+1)} < 0 .$$

$\therefore \quad a_{n+1} - a_n \leqslant 0$ for all n.

Remark. If the sequence (a_n) is a monotone decreasing sequence of nonzero terms, then as discussed above, we have $\frac{a_{n+1}}{an} \leqslant 1$ for all n.

Ex. 5. Prove that the following sequence, whose nth term is given below, is monotonic and bounded.

$$a_n = \frac{2n+3}{n+1}$$

Sol. $\quad a_n = \frac{2n+3}{n+1} .$

$$\therefore \quad a_{n+1} = \frac{2(n+1)+3}{(n+1)+1} = \frac{2n+5}{n+2} .$$

$$\therefore \quad a_{n+1} - a_n = \frac{2n+5}{n+2} - \frac{2n+3}{n+1}$$

$$= \frac{(2n+5)(n+1) - (2n+3)(n+2)}{(n+1)(n+2)}$$

$$= \frac{(2n^2+7n+5)-(2n^2+7n+6)}{(n+1)(n+2)}$$

$$= \frac{-1}{(n+1)(n+2)}$$

$$\therefore \quad a_{n+1}-a_n = \frac{-1}{(n+1)(n+2)} < 0.$$

$$\therefore \quad a_{n+1}-a_n \leqslant 0 \text{ for all } n.$$

$\therefore$ The sequence (a_n) is monotone decreasing. We now show that the sequence (a_n) is bounded.

The sequence (a_n) is bounded below, because for each n,

$$a_n = \frac{2n+3}{n+1} > 0.$$

As the sequence (a_n) is monotone decreasing,

$a_1 \geqslant a_2 \geqslant a_3$ .....

and therefore,

$$a_n \leqslant a_1 \text{ for all } n.$$

Hence (a_n) is bounded above.

i.e. The sequence (a_n) is a bounded sequence.

We now state the following results in respect of the monotonic sequences. We omit their proofs as they are beyond the scope of this book.

Thm. 1. If a monotone increasing sequence is bounded above, then it is convergent.

Thm. 2. If a monotone decreasing sequence is bounded below, then it is convergent.

Further, using above theorems, we can easily deduce the following results.

Thm. 3. If a sequence (a_n) is monotone increasing and bounded, then the set of terms of the sequence has both lub and glb, and in particular

$$\text{lub} = \lim_{n\to\infty} a_n$$

and

$$\text{glb} = a_1.$$

Thm. 4. If a sequence (a_n) is monotone decreasing and bounded, the set of terms of the sequence has both lub and glb, and in particular.

$$\text{lub} = a_1$$

$$\text{and glb} = \lim_{n\to\infty} a_n.$$

We shall now illustrate the above theorem with the help of examples.

Ex. 6. Determine the lub and glb of the set of terms of the sequences whose nth terms are given below.

(*i*) $a_n = \dfrac{1+2+3+\ldots\ldots+n}{n^2}$; (*ii*) $a_n = \dfrac{n+2}{n+3}$

Sol. (*i*) We know that $1+2+3\ldots\ldots+n = \dfrac{n(n+1)}{2}$

$$\therefore \quad a_n = \frac{1+2+3+\ldots+n}{n^2} = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}$$

$$\therefore \quad a_{n+1} = \frac{(n+1)+1}{2(n+1)} = \frac{n+2}{2(n+1)}.$$

$$\therefore \quad a_{n+1}-a_n = \frac{n+2}{2(n+1)} - \frac{n+1}{2n}$$

$$= \frac{n(n+2)-(n+1)^2}{2n(n+1)} = \frac{-1}{2n(n+1)}$$

$\therefore \quad a_{n+1}-a_n < 0$ for all n.

$\therefore$ The sequence (a_n) is monotone decreasing. But $a_n > 0$ for all n. Therefore (a_n) is bounded below. On application of the above theorem, it is evident that (a_n) is monotone decreasing and bounded below and hence

$$\text{glb of the sequence} = \lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} \frac{n+1}{2n}$$

$$= \tfrac{1}{2}.$$

$$\text{and lub of the sequence} = a_1$$

$$= \frac{2}{2} = 1.$$

(*ii*) $a_n = \dfrac{n+2}{n+3}$

$$\therefore \quad a_{n+1} = \frac{(n+1)+2}{(n+1)+3} = \frac{n+3}{n+4}$$

$$\therefore \quad a_{n+1}-a_n = \frac{n+3}{n+4} - \frac{n+2}{n+3}$$

$$= \frac{(n+3)^2-(n+2)(n+4)}{(n+4)(n+3)}$$

$$= \frac{(n^2+6n+9)-(n^2+6n+8)}{(n+4)\,(n+3)}$$

$$= \frac{1}{(n+3)\,(n+4)}$$

$\therefore\ a_{n+1}-a_n = \dfrac{1}{(n+3)\,(n+4)} > 0.$

$\therefore$ (a_n) is monotone increasing.

But $a_n = \dfrac{n+2}{n+3} < 1$ for all n.

A (a_n) is monotone increasing sequence which is bounded above.

$\therefore$ (a_n) is a bounded sequence.

$\therefore$ glb of the sequence $= a_1 = \dfrac{3}{4}$

and lub of the sequence $= \lim\limits_{n\to\infty} a_n.$

$$= \lim_{n\to\infty} \frac{n+2}{n+3}$$

$$= 1.$$

Remark. (*i*) If a monotone increasing sequence is not bounded above, then it is divergent and $\lim\limits_{n\to\infty} a_n = +\infty$.

(*ii*) If a montone decreasing sequence is not bounded below, then it is divergent

and $\lim\limits_{n\to\infty} a_n = -\infty$.

Following examples illustrates above remark.

Ex. 7. Show that the sequences, whose nth terms are given below, are divergent.

(*i*) $a_n = n^2+2n+3$; (*ii*) $a_n = 1+2n-n^2$.

Sol. (*i*) $a_n = n^2+2n+3.$

$\therefore\ a_{n+1} = (n+1)^2+2\,(n+1)+3.$

$= n^2+4n+6.$

$\therefore\ a_{n+1}-a_n = (n^2+4n+6)-(n^2+2n+3).$

$= 2n+3 > 0$

$\therefore$ (a_n) is monotone increasing.

However, the sequence is not bounded above and as per above remark

$$\lim_{n\to\infty} a_n = +\infty.$$

$$(ii) \qquad a_n = 1+2n-n^2.$$

$$\therefore \qquad a_{n+1} = 1+2(n+1)-(n+1)^2.$$

$$= 2-n^2.$$

$$\therefore \quad a_{n+1}-a_n = (2-n^2)-(1+2n-n^2)$$

$$= 1-2n < 0$$

$$\therefore \quad a_{n+1}-a_n < 0 \text{ for all } n.$$

Clearly the sequence (a_n) is monotone decreasing but it is not bounded below. Therefore, as per above remark,

$$\lim_{n\to\infty} a_n = -\infty.$$

EXERCISE 1·4.

1. Give examples of :

(*i*) monotone increasing sequence bounded above,

(*ii*) monotone decreasing sequence not bounded below,

(*iii*) montone decreasing sequence bounded below,

(*iv*) a monotone sequence converging to 5, and

(*v*) monotone increasing sequence which is not bounded above.

2. Show that, the sequences whose nth terms are given below, are monotonic. Determine their lub and glb whenever they exist by making use of the theorems given above.

(*i*) $a_n = 1-2n$; (*ii*) $a_n = 2n+1$; (*iii*) $a_n = \dfrac{3n+1}{n}$;

(*iv*) $a_n = 2-\dfrac{1}{n}$; (*v*) $a_n = \dfrac{1}{(n+1)(n+2)}$;

(*vi*) $a_n = \dfrac{3n+4}{4n+5}$

3. Prove that the sequences, whose nth terms are given below, are monotonic.

(*i*) $a_n = \dfrac{1}{1\cdot2}+\dfrac{1}{2\cdot3}+\dfrac{1}{3\cdot4}+\ldots\ldots+\dfrac{1}{n(n+1)}$

(*ii*) $a_n = \dfrac{1}{n+1}+\dfrac{1}{n+2}+\ldots\ldots+\dfrac{1}{n+n}$

(*iii*) $a_n = \dfrac{(n+1)(n+2)}{(n+3)(n+4)}$

1·11. General Sequences

Now that we have discussed the sequences in detail, we are on the threshold of tackling most of the problems concerned with the sequences. In this article, our intention is to consolidate all the results of the sequences discussed earlier, and to expose the techniques and to illustrate the skilful use of the theorems and the other results through some illustrative examples.

Ex. 1. Prove that, the sequence given by

$$\sqrt{5},\quad \sqrt{5\sqrt{5}},\quad \sqrt{5\sqrt{5\sqrt{5}}},\quad \sqrt{5\sqrt{5\sqrt{5\sqrt{5}}}},\ \ldots\ldots$$

converges to 5.

Sol. Let (a_n) be the given sequence. Then

$$a_1 = \sqrt{5} = 5^{1/2}$$

$$a_2 = \sqrt{5\sqrt{5}} = \left[\ 5^{(1+1/2)}\ \right]^{1/2} = 5^{\left(\frac{1}{2}+\frac{1}{2^2}\right)}$$

$$a_3 = \sqrt{5\sqrt{5\sqrt{5}}} = \left[\ 5^{\left(1+\frac{1}{2}+\frac{1}{2^2}\right)}\ \right]^{\frac{1}{2}}$$

$$= 5^{\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\right)} \text{ etc.}$$

In general,

$$a_n = 5^{\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}\right)}.$$

Now, we know that

$$1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n} = \frac{1}{2}\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right) = 1-\frac{1}{2^n}$$

$$\therefore\ a_n = 5^{\left(1-\frac{1}{2^n}\right)} = 5^{\left[1-\left(\frac{1}{2}\right)^n\right]}$$

$$\therefore\ \text{Also, } \lim_{n\to\infty}\left[1-\left(\frac{1}{2}\right)^n\right] = 1-0 = 1.$$

$$\therefore\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} 5^{[1-(\frac{1}{2})^n]}$$

$$= 5^{\left[\lim_{n\to\infty}\left\{1-\left(\frac{1}{2}\right)^n\right\}\right]} = 5^1 = 5.$$

$\therefore$ The sequence converges to 5.

Ex. 2. Prove that the sequence (a_n), where

$$a_n = \frac{1}{2n+1}+\frac{1}{2n+3}+\frac{1}{2n+5}+\ldots+\ldots+\frac{1}{2n+(2n-1)}$$

converges to a limit lying between $\frac{1}{4}$ and $\frac{1}{2}$.

Sol. We have,

$$a_n = \frac{1}{2n+1} + \frac{1}{2n+3} + \frac{1}{2n+5} + \ldots\ldots + \frac{1}{4n-3} + \frac{1}{4n-1}$$

$$\therefore \quad a_{n+1} = \frac{1}{2n+3} + \frac{1}{2n+5} + \ldots\ldots + \frac{1}{4n-1} + \frac{1}{4n+1} + \frac{1}{4n+3}.$$

$$\therefore \quad a_{n+1} - a_n = \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+1}$$

$$= \frac{4n+1+4n+3}{(4n+1)(4n+3)} - \frac{1}{(2n+1)}$$

$$= \frac{4(2n+1)}{(4n+1)(4n+3)} - \frac{1}{(2n+1)}$$

$$= \frac{4(2n+1)(2n+1) - (4n+1)(4n+3)}{(4n+1)(4n+3)(2n+1)}$$

$$= \frac{(16n^2+16n+4) - (16n^2+16n+3)}{(4n+1)(4n+3)(2n+1)}$$

$$= \frac{1}{(4n+1)(4n+3)(2n+1)} > 0.$$

$\therefore \quad a_{n+1} - a_n > 0$ for all n.

$\therefore \quad (a_n)$ is monotonic increasing.

Observe that, in

$$a_n = \frac{1}{2n+1} + \frac{1}{2n+3} + \frac{1}{2n+5} + \ldots + \frac{1}{4n-1}, \quad \frac{1}{2n+1} \text{ is}$$

the largest and $\frac{1}{(4n-1)}$ is the smallest term. Replacing all the terms of a_n by $\frac{1}{2n+1}$, we get

$$a_n < \frac{1}{2n+1} + \frac{1}{2n+1} + \ldots \text{ to } n \text{ terms.}$$

$$\therefore \quad a_n < \frac{n}{2n+1}. \qquad \ldots(1)$$

Similarly, replacing all the terms of a_n by $\frac{1}{4n-1}$, we get

$$a_n > \frac{1}{4n-1} + \frac{1}{4n-1} \ldots\ldots \text{ to } n \text{ turns.}$$

$$\therefore \quad a_n > \frac{n}{4n-1} \qquad \ldots(2)$$

Combining (1) and (2), we get

$$\frac{n}{4n-1} < a_n < \frac{n}{2n+1} \text{ for all } n. \qquad \ldots(3)$$

Also, $a_n < \frac{n}{2n+1} < \frac{n}{2n} = \frac{1}{2}$.

$\therefore \quad a_n < \frac{1}{2}$.

$\therefore$ (a_n) is monotone increasing and bounded above and therefore it is convergent.

Taking the limits as $n \to \infty$, (3) give us

$$\lim_{n\to\infty}\left(\frac{n}{4n-1}\right) \leqslant \lim_{x\to z} a_n \leqslant \lim_{n\to\infty}\left(\frac{n}{2n+1}\right)$$

i.e. $\frac{1}{4} \leqslant \lim_{n\to\infty} a_n \leqslant \frac{1}{2}$.

Ex. 3. Show that the following sequence

$\cdot 1, \cdot 11, \cdot 111, \cdot 1111, \cdot 11111, \ldots\ldots\ldots$

is convergent.

Sol. Let (a_n) be the given sequence.

Then $a_1 = \cdot 1 = \frac{1}{10}$,

$$a_2 = \cdot 11 = \frac{1}{10} + \frac{1}{10^2},$$

$$a_3 = \cdot 111 = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} \text{ etc.}$$

In general,

$$a_n = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots + \frac{1}{10^n}.$$

$$\therefore \quad a_n = \frac{1}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots + \frac{1}{10^{n-1}}\right)$$

$$= \frac{1}{10}\left[\frac{1-\left(\frac{1}{10}\right)^n}{1-\left(\frac{1}{10}\right)}\right]$$

$$= \frac{1}{10}\left[\frac{1-\left(\frac{1}{10}\right)^n}{\left(\frac{9}{10}\right)}\right]$$

$$= \frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right].$$

$$\therefore \quad a_{n+1} = \frac{1}{9}\left[1-\left(\frac{1}{10}\right)^{n+1}\right].$$

$\therefore\ a_{n+1}-a_n = \frac{1}{9}\left[\left(\frac{1}{10}\right)^n-\left(\frac{1}{10}\right)^{n+1}\right] > 0$

$\therefore\ a_{n+1}-a_n > 0$ for all n.

$\therefore\ (a_n)$ is monotone increasing.

But $a_n < 1$ for all n.

$\therefore$ The sequence (a_n) is monotone increasing and bounded above, and hence it is convergent.

Now, $a_n = \frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]$.

$\therefore$ Taking the limits as $n\to\infty$, we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{\frac{1}{9}\left[1-\left(\frac{1}{10}\right)^n\right]\right\}$$
$$= \frac{1}{9}.$$

$\therefore$ The sequence (a_n) converges to $\left(\frac{1}{9}\right)$.

EXERCISE 1·5.

1. Prove that the following sequences are bounded and monotonic. Hence or otherwise obtain their limits.

(i) $\sqrt{2}$, $\sqrt{2\sqrt{2}}$, $\sqrt{2\sqrt{2\sqrt{2}}}$, $\sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}$, ... ,.....

(ii) $\sqrt{7}$, $\sqrt{7\sqrt{7}}$, $\sqrt{7\sqrt{7\sqrt{7}}}$, $\sqrt{7\sqrt{7\sqrt{7\sqrt{7}}}}$, ...

(iii) $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, ...

(iv) $\sqrt{3}$, $\sqrt{3+\sqrt{3}}$, $\sqrt{3+\sqrt{3+\sqrt{3}}}$, ...

(v) 0·2, 0·22, 0·222, 0·2222, ...

(vi) 0·7, 0·77, 0·777, 0·7777, ...

2. Prove that the following sequences, whose nth terms are given below, are convergent.

(i) $a_n = \frac{1}{3n+1}+\frac{1}{3n+2}+\frac{1}{3n+3}+\cdots\cdots+\frac{1}{3n+n}$.

(ii) $a_n = \frac{1}{4n-1}+\frac{1}{4n-2}+\frac{1}{4n-3}+\cdots\cdots+\frac{1}{4n-n}$

(iii) $a_n = \frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots\cdots+\frac{1}{\sqrt{n^2+n}}$

3. Prove that the sequence (a_n) where

$a_n = \frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\cdots\cdots+\frac{1}{\sqrt{n+n}}$ is monotone increasing and diverges to $+\infty$.

4. Prove that the sequence (a_n) in which

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots\ldots + \frac{1}{n+n}$$ is a bounded monotone sequence and show that it converges to a limit lying between $\frac{1}{2}$ and 1.

5. Prove that the sequences, whose nth terms are given below, are convergent.

(i) $a_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots\ldots\ldots + \frac{1}{(n+n)^2}$

(ii) $a_n = \frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} + \ldots\ldots\ldots + \frac{1}{(n+n)^3}$

6. Show that both the sequences in the above example converge to O.

Answers and Hints

1. (i) 2 ; (ii) 7 ; (iii) Positive root of $x^2-x-2=0$ i.e. 2 ; (iv) Positive root of $x^2-x-3=0$; (v) $\frac{2}{9}$; (vi) $\frac{7}{9}$.

2. (i) $a_n < \frac{n}{3n+1} < \frac{1}{3}$; (a_n) is bounded.

(ii) $a_n < \frac{n}{4n-n} = \frac{1}{3}$; (a_n) is bounded.

(iii) $a_n < \frac{n}{(n+1)^2} < 1$ etc.

3. Hint : $a_n > \frac{n}{\sqrt{n+n}} = \frac{\sqrt{n}}{2}$,

$\therefore$ a_n diverges to $+\infty$.

4. Hint : $\frac{n}{n+n} < a_n < \frac{n}{n+1}$.

$\therefore$ $\frac{1}{2} < a_n < 1-\frac{1}{n+1}$ etc.

5. Hints : (i) $a_n < \frac{n}{(n+1)^2}$ etc.

(ii) $a_n < \frac{n}{(n+1)^3}$ etc.

6. Hints : (i) $\frac{n}{(n+n)^2} < a_n < \frac{n}{(n+1)^2}$

(ii) $\frac{n}{(n+n)^3} < a_n < \frac{n}{(n+1)^3}$

1·12. The Number 'e'

We are now in position to discuss the convergence of an important sequence. It is the sequence (U_n) where,

$$U_n = \left(1+\frac{1}{n}\right)^n$$

We shall prove that it is a bounded monotonic increasing sequence and hence it converges.

First we show that the sequence (Un) is monotone increasing.

By the binomial theorem for the positive integral indices we have

$$U_n = \left(1+\frac{1}{n}\right)^n$$

$$= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \ldots + \frac{1}{n^n}$$

$$= 1+1+\frac{1-\frac{1}{n}}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)}{n!}, \quad \ldots (1)$$

and,

$$U_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$= 1+(n+1)\frac{1}{(n+1)} + \frac{(n+1)(n+1-1)}{2!}\cdot\frac{1}{(n+1)^2} +$$

$$\ldots\ldots + \frac{1}{(n+1)^{n+1}}$$

$$= 1+1+\frac{1-\frac{1}{n+1}}{2!} + \frac{\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)}{3!} + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots + \frac{\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\left(1-\frac{n+1-1}{n+1}\right)}{(n+1)!}, \ldots\ldots \quad (2)$$

By inspecting (1) and (2), we observe that (*i*) the first two terms of (1) and (2) are equal and in the remaining terms, each term of (1) is less than the corresponding term of (2), (*ii*) there are $(n+2)$ terms in (2) and $n+1$ terms in (1) and (*iii*) all the terms in (1) and (2) are positive. Therefore it follows that,

$U_{n+1} > U_n$ for all n.

$\therefore$ The sequence is monotonic increasing.

Now we show that the sequence (Un) is bounded.

Since the sequence is monotone increasing, it is bounded below. We shall now prove that it is also bounded above.

We have, from (1)

$$U_n = 1+1+\frac{1-\frac{1}{n}}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{n-1}{n}\right)}{n!}$$

$$\therefore \quad U_n < 1+1+\frac{1}{2!}+\frac{1}{3!}\ldots\ldots+\frac{1}{n!} \qquad \ldots(3)$$

Now consider the sequence (t_n) where

$$t_n = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\ldots\ldots\ldots+\frac{1}{n!} \qquad \ldots(4)$$

It is evident that,

$$\frac{1}{2!} = \frac{1}{2}\,;\ \frac{1}{3!} = \frac{1}{1\cdot2\cdot3} < \frac{1}{1\cdot2\cdot2} = \frac{1}{2^2}\,;$$

$$\frac{1}{4!} = \frac{1}{1\cdot2\cdot3\cdot4} < \frac{1}{1\cdot2\cdot2\cdot2} = \frac{1}{2^3} \ldots\ldots \text{etc.}$$

In general,

$$\frac{1}{r!} = \frac{1}{1\cdot2\cdot3\ldots\ldots r} < \frac{1}{1\cdot2\cdot2\ldots\ldots(r-1)\text{ factors}}$$

$$\therefore \quad \frac{1}{r!} < \frac{1}{2^{r-1}}\,.$$

$$\therefore \quad t_n < 1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots+\frac{1}{2^{n-1}}\right)$$

$$= 1+\frac{\left[1-\frac{1}{2^n}\right]}{\left[1-\frac{1}{2}\right]} = 3-\frac{1}{2^{n-1}}$$

$$\therefore \quad t_n < 3-\frac{1}{2^{n-1}} < 3.$$

$\therefore \quad t_n < 3$ for all n.

Now we have from (3),

$U_n < t_n < 3$

$U_n < 3$ for all n.... ... (5)

$\therefore$ The sequence (U_n) is bounded above.

$\therefore$ (U_n) is a bounded monotonic sequence.

$\therefore$ The sequence (U_n) converges to a limit. We denote this limit by e.

$$\therefore \quad \lim_{n\to\infty} U_n = e.$$

$$\therefore \quad \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

Now, from (1), we have,

$U_n > 2$, for all n.

$\therefore \quad 2 < U_n < 3$

By taking limits, we get

$$2 \leqslant \lim_{n\to\infty} U_n \leqslant 3$$

$\therefore \quad 2 \leqslant e \leqslant 3.$

It can be proved that e is an irrational number and its value correct up to 6 decimal places is.

$$e = 2{\cdot}718281.$$

Remark

In the course of the above discussion, we have considered the sequence (t_n) where,

$$t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!},$$

Clearly, $t_{n+1} = 1 + \frac{1}{1!} + \frac{1}{2!} - \ldots\ldots + \frac{1}{n!} + \frac{1}{(n+1)!}$.

$\therefore \quad t_{n+1} - t_n = \frac{1}{(n+1)!} > 0$ for all n.

$\therefore \quad t_{n+1} > t_n$ for all n.

$\therefore$ The sequence (t_n) is monotonic increasing. Also we have proved that it is bounded above.

$\therefore$ The sequence is convergent.

It can be proved that the limit of this sequence is the same as $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ *i.e.*, the number e.

Thus we have,

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!}\right).$$

1·13. Infinite Series

Uptil now, we discussed the behaviour of the nth term of a sequence where n tends to ∞. We observed that if (a_n) and (b_n) are two sequences, then (ka_n), $(a_n + b_n)$, $(a_n - b_n)$, $(a_n \cdot b_n)$ and (a_n/b_n) are some of the sequences that can be obtained with the help of the sequences (a_n) and (b_n). There is another way by means of which we can obtain a new sequence which has not been discussed hitherto. Let (a_n) be a sequence given. For each n, define s_n as the sum of the first n terms of the sequence (a_n). Clearly (s_n) is a new sequence obtained from (a_n). It is interesting to discuss the behaviour of the nth term of the sequence (s_n) when n becomes larger and larger.

One could straightway ask why we do not add up all the terms of the sequence (a_n) and discuss rather than by the roundabout method proposed above. We shall first answer this question before going ahead with the methodical development of the topic.

More often that not, the results which are true for a finite number of terms in an expression go wrong for infinitely many terms. To illustrate this phenomenon, we consider an expression containing infinitely many terms.

$1-1+1-1+1-1+1-1+1-1+1-1$ .....

We can write it as

$(1-1)+(1-1)+(1-1)+(1-1)+(1-1)+$ ..... which means, we have

$0+0+0+0$ ..... $+ 0$ ..... $= 0.$

But, on the other hand, we can write the same expression as

$1-(1-1)-(1-1)-(1-1)-(1-1)$ .....

which is the same as

$1-0-0-0-0-0$ .....$= 1.$

This implies that the same expression stands for 1 as well as 0 and this is contradictory to our assumptions about the system of real numbers. This is a clear indication that while dealing with infinitely many terms, we are likely to violate very innocently some rules and thereby get results for which we are totally unprepared. There are two ways to avoid this embarrassing situation—first one is not to deal at all with infinitely many terms, which is rather unpleasant; and the second one is to deal with infinitely many terms through the idea of limits. For this very reason, we are adopting the *circuitous route* mentioned earlier.

We begin the discussion of this topic with a few definitions.

Let (a_n) be any sequence. Then the sum of the terms of (a_n)

$a_1 + a_2 + a_3 + a_4 +$ .... (A)

is denoted by $\sum_{n=1}^{\infty} a_n$, symbolically. We call $\sum_{n=1}^{\infty} a_n$ *an infinite series*. Therefore, an infinite series is sum of the terms of a sequence. However, the sum may not be defined as illustrated earlier by an example. But we can always obtain the sum of the finite number of the terms of a sequence.

Let

$s_n = a_1 + a_2 + a_3 +$ ..... $+a_n$, ... (B)

We call s_n *the partial sum of order n* of the infinite series Σa_n.

Clearly, $s_1 = a_1$

$s_2 = a_1 + a_2$,

$s_3 = a_1 + a_2 + a_3$, ... etc.

Consider now the following series :

(*i*) $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots + \frac{1}{2^{n-1}} + \ldots\ldots$

(*ii*) $1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots + n^2 + \ldots\ldots$

(*iii*) $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots\ldots + \frac{1}{n^3} + \ldots\ldots$

(*iv*) $1-1+1-1+1-1+-1-\ldots\ldots + (-1)^{n-1}\ldots\ldots$

Clearly, for the series (*i*) above,

$$s_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{n-1}}$$

$$= 2\left(1 - \frac{1}{2^n}\right).$$

For the series (*ii*) above,

$$s_n = 1^2 + 2^2 + \ldots\ldots + n^2$$

$$= \frac{n(n+1)(2n+1)}{6}.$$

For the series (*iii*) above, we cannot obtain a simplified expression for s_n. We simply write,

$$s_n = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots\ldots + \frac{1}{n^3}.$$

For the series (*iv*),

$$s_n = \begin{cases} 0 \text{ if } n \text{ is even number.} \\ 1 \text{ if } n \text{ is odd number.} \end{cases}$$

Convergence of infinite series

Let us consider the behaviour of the sequences (s_n) of the partial sums of the series (*i*), (*ii*) and (*iii*) given above.

For the series (*i*), we have

$$s_n = 2\left(1 - \frac{1}{2_n}\right).$$

$$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[2\left(1 - \frac{1}{2^n}\right)\right]$$

$$= \lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{1}{2^{n-1}}$$

$$= 2.$$

∴ The sequence of the partial sums of the series $\sum \frac{1}{2^{n-1}}$ converges to 2.

Similarly, for the series (*ii*), we have

$$s_n = \frac{n\,(n+1)\,(2n+1)}{6}$$

$$= \tfrac{1}{6}\,[\,2n^3+3n^2+n\,].$$

Now, $\lim_{n\to\infty} n^3 = +\infty$,

$\lim_{n\to\infty} n^2 = +\infty$ and

$\lim_{n\to\infty} n = +\infty$.

$\therefore \lim_{n\to\infty} s_n = +\infty$.

The sequence of the partial sums of the series (*ii*) diverges to $+\infty$.

For the series (*iii*), we cannot draw any conclusion for the time being as no simplified expression for the partial sum s_n is available.

The partial sum of the series (*iv*) oscillates finitely as n tends to $+\infty$.

Def. 1. The infinite series $\sum_{n=1}^{\infty} a_n$ is said *to be convergent* if the sequence (s_n) of the partial sums

$s_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n$, converges.

In this case, $\lim_{n\to\infty} s_n$ is called the *sum* of the infinite series.

In the same way, if the sequence (s_n) diverges to $+\infty$, then the infinite series $\Sigma\, a_n$ is said to *diverge to* $+\infty$ and if the sequence (s_n) *diverges to* $-\infty$, then the infinite series $\Sigma\, a_n$ is said to diverge to $-\infty$. Finally, if the sequence (s_n) oscillates (finitely or infinitely), then the infinite series $\Sigma\ a_n$ is said to oscillate (finitely or infinitely).

Let us consider now a few examples to discuss the convergence behaviour of infinite series.

Ex. 1. Discuss the convergence behaviour of the infinite series $\Sigma\, 2^n$.

Sol. We have,

$$s_n = 2 + 2^2 + 2^3 + \ldots\ldots + 2^n$$

$$= 2\,(1 + 2 + 2^2 + \ldots\ldots\ 2^{n-1})$$

$$= \frac{2\,(2^n-1)}{2-1} = 2\,(2^{n-1})$$

$$= 2^{n+1} - 2.$$

Clearly $2^{n+1} \longrightarrow +\infty$ as $n \longrightarrow \infty$.

$\therefore$ $s_n \longrightarrow +\infty$ as $n \longrightarrow \infty$.

$\therefore$ The sequence (s_n) diverges to $+\infty$.

$\therefore$ The infinite series $\Sigma\, 2^n$ diverges to $+\infty$.

Ex. 2. Discuss the convergence of the infinite series $\Sigma\,(-\frac{1}{3})^{n-1}$.

Sol. We have,

$$s_n = 1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \ldots\ldots + \left(-\frac{1}{3}\right)^{n-1}$$

$$= \frac{1-\left(-\frac{1}{3}\right)^n}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{3}{4}\left[1-\left(-\frac{1}{3}\right)^n\right]$$

Now, $\lim\limits_{n\to\infty} (-\frac{1}{3})^n = 0$.

$\therefore$ $\lim\limits_{n\to\infty} s_n = \frac{3}{4}$.

The infinite series $\sum\limits_{n=1}^{\infty}\left(-\frac{1}{3}\right)^{n-1}$ converges to $\frac{3}{4}$ and consequently, $\frac{3}{4}$ is the sum of this series.

Convergent Infinite Series

The convergence of an infinite series is the same as the convergence of the sequence of its partial sums. Therefore, the problem of convergence of an infinite series is very similar to the problem of convergence of a sequence. However, there are infinite series for which it is not easy and sometimes impossible, to obtain a simplified expression for their partial sum. Therefore, even to discuss the behaviour of the partial sum becomes difficult. Under the circumstances, we shall have to curtail our desire to know whether the series has a sum or not and what is the sum of the series if it exists ; and be contented with the information whether the infinite series converges, diverges or oscillates.

Due to limitations of the scope of this book, we classify the infinite series into two distinct types and deal with them step by step.

We consider following two types :

Type (A). The infinite series for which a simple expression for the partial sum can be obtained with the help of some standard procedures. Certainly in this case we shall be in position to determine convergence behaviour of the infinite series and in addition to this we shall be in position to obtain the sum of the series whenever it exists.

Type (B). The infinite series of positive terms to which some simple standard tests of convergence can be applied to *just determine* whether the series under consideration is convergent or divergent. These tests do not reveal any information about the sum of the series in case it exists.

1·14. Series of the type A

Whenever we can obtain a simple expression for the partial sums, we include those series in type A. We can certainly decide for the series of type A whether it converges or diverges and if it converges, then what is its sum. However there are different classes even in type A and we shall consider them one by one.

(I) Geometric Series

The simplest series in type A is the *geometric series* Σa_n where the sequence (a_n) is a geometric sequence.

We are aware that if $a_n = x^n$, then

for $|x| < 1$, $\lim_{n\to\infty} a_n = 0$ and

for $x > 1$, $\lim_{n\to\infty} a_n = \infty$

Let us first consider some examples.

Ex. 1. Discuss the convergence of $\Sigma\ (-\frac{2}{3})^{n-1}$.

Sol. Let S_n be the partial sum of order n for the series $\Sigma(-\frac{2}{3})^{n-1}$. Then,

$$S_n = \frac{1-(-\frac{2}{3})^n}{1-(-\frac{2}{3})} = \tfrac{3}{5}\,[1-(-\tfrac{2}{3})^n\,]$$

Now, $|-\frac{2}{3}| = \frac{2}{3} < 1$ and therefore

$$\lim_{n\to\infty} \left(-\frac{2}{3}\right)^n = 0$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \tfrac{3}{5}\,[1-(-1\tfrac{2}{3})^n]$$

$$= \tfrac{3}{5}.$$

$\therefore$ The sequence (S_n) of partial sums converges to $\frac{5}{3}$,

$\therefore$ The infinite series $\Sigma\ (-\frac{2}{3})^{n-1}$ is convergent and its sum is $\frac{3}{5}$.

Ex. 2. Discuss the convergence of the infinite series

$$\Sigma\, 3(\tfrac{3}{2})^{n-1}.$$

Sol. Let S_n be the partial sum of order n for the series under consideration.

$$\therefore \quad S_n = \frac{3\,[(\frac{3}{2})^n - 1]}{(\frac{3}{2}-1)}$$

$$= 6\,[(\tfrac{3}{2})^n - 1].$$

Now, $|\,\frac{3}{2}\,| > 1$ and therefore

$$\lim_{n\to\infty} (\tfrac{3}{2})^n = +\infty$$

$$\therefore \quad \lim_{n\to\infty} S_n = +\infty$$

$\therefore$ The sequence (S_n) of partial sums diverges to $+\infty$ and consequently the infinite series $\Sigma\, 3\,(\frac{3}{2})^{n-1}$ diverges to $+\infty$.

Now, let us consider the general infinite geometric series $\Sigma a\, x^{n-1}$, $a > 0$. Let S_n be the partial sum of order n for this series. Then, wehave

$$S_n = \begin{cases} \dfrac{a\,(1-x^n)}{(1-x)} = \dfrac{a\,(x^n-1)}{(x-1)} & ;\ \text{if } x \neq 1 \\ na & ;\ \text{if } x = 1. \end{cases}$$

Case (*i*). Let $|\,x\,| < 1$. Then $-1 < x < 1$.

Clearly $\lim_{n\to\infty} x^n = 0$.

$$\therefore \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{a\,(1-x^n)}{(1-x)}\right]$$

$$= \lim_{n\to\infty} \left[\frac{a}{1-x} - \frac{a}{1-x}\, x^n\right]$$

$$= \frac{a}{1-x}.$$

$\therefore$ The infinite geometric series $\Sigma\, ax^n$ converges when $-1 < x < 1$ and its sum is $\dfrac{a}{(1-x)}$.

Case (*ii*). Let $x > 1$.

Then, $\lim_{n\to\infty} x^n = +\infty$

$$\therefore \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{a\,(x^n-1)}{(x-1)}\right]$$

$$= \lim_{n\to\infty} \left[\frac{a}{x-1}\, x^n - \frac{a}{x-1}\right]$$

$$= +\infty$$

$\therefore$ The sequence of partial sums (S_n) diverges to $+\infty$.

$\therefore$ The infinite series $\Sigma\, a\, x^{n-1}$ diverges to $+\infty$ when $x > 1$.

Case (*iii*). Let $x < -1$.

Then the sequence (x^n) oscillates infinitely. Clearly the sequence (S_n) also oscillates infinitely; and therefore the infinite geometric series $\Sigma\, ax^{n-1}$ oscillates infinitely when $x < -1$.

Case (*iv*). For $x = -1$, it can be similarly shown that the infinite series Σax^{n-1} oscillates finitely.

Case (*v*). When $x = 1$, $S_n = na$.

It is evident that

$$\lim_{n\to\infty} S_n = +\infty$$

$\therefore$ The infinite geometric series $\Sigma\, ax^{n-1}$ diverges to $+\infty$ when $x = 1$.

Hence we can conclude that the infinite geometric series $\Sigma\, ax^{n-1}$; $a > 0$.

(*i*) converges to $\frac{a}{1-x}$ if $|x| < 1$,

(*ii*) diverges to $+\infty$ if $x \geqslant 1$,

(*iii*) oscillates finitely if $x = -1$,

and (*iv*) oscillates infinitely if $x < -1$.

Ex. 3. Obtain the sum of the infinite series $\Sigma\, 4\,(\frac{1}{3})^{n-1}$.

Sol. The infinite series is a geometric series of the type $\Sigma\, ax^{n-1}$ where $a = 4$ and $x = \frac{1}{3}$. The sum of the geometric series, when $|x| < 1$, is $\frac{a}{1-x}$.

$\therefore$ The sum of the infinite series $\Sigma\, 4\,(\frac{1}{3})^{n-1}$ is

$$\frac{4}{1-\frac{1}{3}} = \frac{4}{(\frac{2}{3})} = 6.$$

(II) A very simple expression for the partial sum of an infinite series can also be obtained for certain series other than the geometric series. The following examples illustrate this class of series and also the techniques used to deal with them.

Ex. 4. Obtain the sum of the series $\Sigma\, \frac{1}{n\,(n+1)}$.

Sol. Let the given series be denoted by $\Sigma\, a_n$.

Then, clearly $a_n = \frac{1}{n\,(n+1)}$.

We can express

$$a_n = \left(\frac{1}{n} - \frac{1}{n+1}\right) \text{ for all } n.$$

Now, let S_n be the partial sum of order n for the series under consideration.

Then, $S_n = a_1 + a_2 + \ldots\ldots\ldots + a_n$

$$= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots\ldots\ldots + \frac{1}{n(n+1)}$$

$$= \left[1 - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \ldots\ldots + \left[\frac{1}{n} - \frac{1}{n+1}\right].$$

It can be seen that except for the first bracket, the first term in every bracket cancels with the second term of the previous bracket.

$$\therefore \quad S_n = 1 - \frac{1}{n+1}$$

$$\therefore \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

$\therefore$ The sum of the infinite series $\Sigma \dfrac{1}{n(n+1)}$ is 1.

Ex. 5. Find the sum of the infinite series

$$\frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \frac{1}{10\cdot 13} + \ldots\ldots + \frac{1}{(3n+1)(3n+4)} + \ldots.$$

Sol. Let Σa_n be the infinite series under consideration. Then,

$$a_n = \frac{1}{(3n+1)(3n+4)} \text{ for all } n.$$

We can express a_n as

$$a_n = \frac{1}{3}\left[\frac{1}{3n+1} - \frac{1}{3n+4}\right] \text{ for all } n.$$

Let (S_n) be the sequence of the partial sums for the series under consideration.

We have, then

$$S_n = a_1 + a_2 + \ldots\ldots + a_n$$

$$= \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \frac{1}{10\cdot 13} + \ldots\ldots + \ldots\ldots$$

$$\ldots\ldots\ldots + \frac{1}{(3n+1)(3n+4)}$$

$$= \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right) + \frac{1}{3}\left(\frac{1}{7} - \frac{1}{10}\right) + \frac{1}{3}\left(\frac{1}{10} - \frac{1}{13}\right)$$

$$\ldots\ldots\ldots + \ldots\ldots + \frac{1}{3}\left(\frac{1}{3n+1} - \frac{1}{3n+4}\right)$$

$$= \frac{1}{3}\left[\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\left(\frac{1}{10}-\frac{1}{13}\right)+ \right.$$
$$\left. \ldots\ldots + \left(\frac{1}{3n+1}-\frac{1}{3n+4}\right)\right]$$
$$= \frac{1}{3}\left(\frac{1}{4}-\frac{1}{3n+4}\right)$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3n+4}\right)\right] = \frac{1}{12}.$$

$\therefore$ The sum of the infinite series is $\frac{1}{12}$.

Observe that in both the examples given earlier, denominator of each term of the series is a product of two consecutive terms of an arithmetic sequence.

In case of Ex. 5, the denominators of the terms can be written as

1·2, 2·3, 3·4, 4·5, 5·6, ... $n\,(n+1)$ and it can be seen that each term written above is the product of the two consecutive terms of the arithmetic sequence

1, 2, 3, 4, 5, 6, 7, 8,.....n, $(n+1)$, ...

Same is the case of Ex. 6.

i.e. The denominator of the first term of the series is the product of the first and the second term of the arithmetic sequence, the denominator of the second term of the series is the product of the second and the third term of the arithmetic sequence and so on. In general, the denominator of the nth term of the series is the product of the nth term and $(n+1)^{th}$ term of the arithmetic sequence.

In such cases, it is advisable to express each term as the difference of two terms to simplify the expression for the partial sum of order n.

A general formula can be given as follows :

Let $\Sigma\, a_n$ be an infinite series such that $a_n = \dfrac{1}{(b_n)\,(b_{n+1})}$ where (b_n) is an arithmetic progression with common difference d, then

$$a_n = \frac{1}{d}\left[\frac{1}{b_n} - \frac{1}{b_{n+1}}\right] \text{ for all } n.$$

It is for this reason, we could write

$$a_n = \frac{1}{n\,(n+1)} = \frac{1}{1}\cdot\left[\frac{1}{n} - \frac{1}{n\quad 1}\right] \text{ in Ex. 5, and}$$

$$a_n = \frac{1}{(3n+1)\,(3n+4)} = \frac{1}{3}\left[\frac{1}{3n+1} - \frac{1}{3n+4}\right] \text{ in Ex. 6.}$$

There are also some other infinite series for which it is easier to obtain the sum by slightly adjusting each term as we have done in the above examples. It may be noted, however, that the principle is to express each term as the difference of two suitable terms.

Ex. 7. Obtain the sum of the infinite series $\sum \frac{n}{(n+1)!}$.

Sol. Let the infinite series be denoted as Σa_n.

Clearly $a_n = \frac{n}{(n+1)!}$

Now, we have

$$a = \frac{n}{(n+1)!} = \frac{(n+1)-1}{(n+1)!}$$

$$= \frac{(n+1)}{(n+1)!} - \frac{1}{(n+1)!}$$

$$= \left[\frac{1}{n!} - \frac{1}{(n+1)!}\right].$$

Let S_n be the partial sum of order n for this infinite series. Then,

$$S = a_1 + a_2 + \ldots\ldots\ldots + a_n$$

$$= \left[\frac{1}{1!} - \frac{1}{2!}\right] + \left[\frac{1}{2!} - \frac{1}{3!}\right] + \left[\frac{1}{3!} - \frac{1}{4!}\right]$$

$$+ \ldots\ldots + \ldots\ldots\ldots + \left[\frac{1}{n!} - \frac{1}{(n+1)!}\right]$$

$$= 1 - \frac{1}{(n+1)!}$$

Now, note that

$$\lim_{n\to\infty} \frac{1}{(n+1)!} = 0$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[1 - \frac{1}{(n+1)!}\right] = 1.$$

$\therefore$ The sequence (S_n) of partial sums converges to 1 and therefore the infinite series converges to 1.

$\therefore$ The sum of the infinite series is 1.

EXERCISE 1

1. Prove that the following series are convergent and find their sums.

(*i*) $\frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \frac{1}{4\cdot5} \ldots\ldots\ldots + \frac{1}{(n+1)(n+2)} + \ldots\ldots\ldots$

(*ii*) $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots\ldots\ldots + \frac{1}{(2n-1)(2n+1)} + \ldots\ldots$

(iii) $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\ldots\ldots+\frac{1}{n(n+2)}+\ldots\ldots\ldots\ldots$

(iv) $\frac{1}{5\cdot 9}+\frac{1}{9\cdot 13}+\frac{1}{13\cdot 17}+\ldots\ldots\ldots+\frac{1}{(4n+1)(4n+5)}+\ldots\ldots$

(v) $\sum \frac{1}{n^2+13n+42}$

(vi) $\sum \frac{2n+1}{(n^2+2)(n^2+2n+3)}$

(vii) $\sum \frac{2n+1}{(2n^2+1)[2(n+1)^2+1]}$

(iii) $\sum \frac{(-1)^{n+1}(2n+1)}{n(n+1)}$

(ix) $\sum \frac{(-1)^{n+1}n}{4n^2-1}$

(x) $\sum_{n=1}^{\infty} \frac{n}{(n+2)!}$ (Hint : Use definition of e).

(xi) $\sum_{n=1}^{\infty} \frac{n+1}{(n+2)!}$

ANSWERS

(i) $\frac{1}{2}$; (ii) $\frac{1}{2}$; (iii) $\frac{3}{4}$; (iv) $\frac{1}{20}$; (v) $\frac{1}{7}$; (vi) $\frac{1}{3}$;

(vii) $\frac{1}{3}$; (viii) 1 ; (ix) $\frac{1}{4}$; (x) 3—e (xi) $\frac{1}{2}$.

1·15. Series of type B

We shall consider only the series of positive terms in this type. Our intention is to develop a procedure to determine whether the series of this type are convergent or divergent. There are some simple tests to find out whether a given series converges by comparing the given series with some known standard series whose convergence property is already known. However, these tests do not indicate how the sum of a given series is to be obtained in case it is known to be convergent. We shall have to get acquainted with some standard tests as well as some standard series. We begin with some standard tests.

Comparison Tests

Let Σa_n and Σb_n be any two series of non-negative terms and let

$$0 \leqslant a_n \leqslant b_n \text{ for all } n \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Test (1). If the series $\Sigma\, b_n$ is known to be convergent, then the series $\Sigma\, a_n$ is also convergent.

Test (2). If the series $\Sigma\, a_n$ is known to be divergent, then the series $\Sigma\, b_n$ is also divergent.

Remark. Let Σa_n and Σb_n be series of non-negative terms satisfying (1). If $\Sigma\, b_n$ is known to be diverrgent, then it *does not imply* that $\Sigma\, a_n$ is divergent and if $\Sigma\, a_n$ is known to be convergent, then it *does not imply* that $\Sigma\, b_n$ is convergent.

We shall illustrate these comparison tests and above remark after discussing some standard series.

A Standard series for comparison.

Thm. The series $\Sigma \frac{1}{n^p}$ is

divergent if $p \leqslant 1$ and convergent if $p > 1$.

Proof. (*i*) Let $p = 1$. Then the series is

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots\ldots\ldots + \frac{1}{n} + \ldots\ldots\ldots$$

Let S_n be the partial sum for this series.

Then,

$$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots\ldots\ldots + \frac{1}{n}.$$

Now, observe that

$$S_2 = 1 + \tfrac{1}{2} = \tfrac{2}{2},$$

$$\begin{aligned} S_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} \\ &= 1 + (\tfrac{1}{2}) + (\tfrac{1}{3} + \tfrac{1}{4}) \\ &> 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} = 2. \end{aligned}$$

Or

$$S_4 > 2.$$

Also,

$$\begin{aligned} S_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} \\ &= 1 + (\tfrac{1}{2}) + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + + \tfrac{1}{7} + \tfrac{1}{8}) \\ &> 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{5}{2}. \end{aligned}$$

$$\therefore\ S_8 > \tfrac{5}{2}.$$

In S_8, we have grouped the first term, the second term, the third and the fourth and finally from the 5th to the 8th term together. Observe that in the first bracket, there is only one term, in the second bracket there are $2 = 2^1$ terms, in the third bracket there are

$2^2 = 4$ terms. Also, for obtaining the inequalities, we have replaced each term of the bracket by the smallest term of that bracket.

We rewrite the earlier results as follow :

$$S_{2^1} = S_2 > 1 + \frac{10}{2},$$

$$S_{2^2} = S_4 > 1 + \frac{2}{2} \text{ and}$$

$$S_{2^3} = S_8 > 1 + \frac{3}{2}$$

Generalizing the above results, the partial sum of first 2^m terms of the series is given by,

$$S_{2^m} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} \ldots\ldots\ldots + \frac{1}{2^m}$$

$$= 1 + (\tfrac{1}{2}) + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8})$$

$$+ \ldots\ldots\ldots + \left(\frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2)} \ldots\ldots + \frac{1}{2^m}\right)$$

Now replacing each term of the bracket by the smallest term of that bracket (which is the last term of that bracket), we have

$$S_{2^m} > 1 + (\tfrac{1}{2}) + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) +$$

$$+ \ldots\ldots + \left(\frac{1}{2^m} + \frac{1}{2^m} + \ldots\ldots \text{ to } 2^{m-1} \text{ term}\right)$$

It is clear that each bracket in the above expression on simplification becomes $\frac{1}{2}$, and in all there are m such brackets.

$\therefore \quad S_{2^m} > 1 + m\,(\tfrac{1}{2})$

Since $1 + \frac{m}{2} \longrightarrow \infty$ as $m \longrightarrow \infty$, it follows that

$S_{2^m} \longrightarrow \infty$ as $m \longrightarrow \infty$

$\therefore$ We have proved that $S_n \longrightarrow \infty$ as $n \longrightarrow \infty$

where $n = 2^m$

Now, let n be any positive integer.

We can find n such that $n > 2^m$

$\therefore \quad S_n > S_{2^m}$ since (S_n) is a monotone increasing sequence.

$\therefore$ by test (2) (S_n) diverges to $+\infty$

$\therefore \quad \sum \frac{1}{n}$ diverges to $+\infty$

(*ii*) Let $p < 1$.

Then $n^p < n$ for all n.

$\therefore \quad \frac{1}{n^p} > \frac{1}{n}$ for all n.

Since $\Sigma \frac{1}{n}$ is divergent by (*i*) above, by the comparison test (2), t

$\Sigma \frac{1}{n^p}$ is also divergent.

(*iii*) Let $p > 1$. Then we have,

$$S_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots\ldots + \frac{1}{n^p}.$$

Now, $S_1 = 1$,

$$S_2 = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right).$$

$$< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) = 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}.$$

$$\therefore \quad S_3 < 1 + \frac{1}{2^{p-1}}.$$

Similarly,

$$S_7 = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$$

$$< 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$$

$$= 1 + \frac{2}{2^p} + \frac{4}{4^p}.$$

$$\therefore \quad S_7 = 1 + \frac{2}{2^p} + \frac{4}{4^p}.$$

Now, we can write $1 = 2 - 1$, $3 = 2^2 - 1$, $7 = 2^3 - 1$ etc.

Therefore, we rewrite above results as

$$S_1 = S_{2-1} = 1$$

$$S_3 = S_{2^2-1} < 1 + \frac{2}{2^p}$$

$$S_7 = S_{(2^3-1)} < 1 + \frac{2}{2^p} + \frac{4}{4^p}$$

To obtain the above inequalities we have grouped the terms of the corresponding partial sums in such a way that

(*i*) only the terms appearing after the first have been put into brackets,

(*ii*) the first bracket has 2 terms, the second has 2^2 terms and

(*iii*) each term of the bracket has been replaced by the largest term of that bracket.

Generalizing the above principle, we have

$$S_{(2^m-1)} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{(2^m-1)^p}$$

$$= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots$$

$$+ \ldots\ldots + \left[\frac{1}{(2^{m-1})^p} + \frac{1}{(2^{m-1}+1)^p} + \ldots\ldots + \right.$$

$$\left. \ldots\ldots\ldots + \frac{1}{(2^m-1)^p}\right]$$

On replacing each term of the bracket by the large $\Rightarrow$ term of that bracket (which happens to be the very first term of each bracket), we get

$$S_{2^m-1} < 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \ldots$$

$$\ldots\ldots + \ldots\ldots + \left[\frac{1}{(2^{m-1})^p} + \frac{1}{(2^{m-1})^p} + \ldots \text{ to } 2^{m-1} \text{ terms}\right]$$

$$= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \ldots\ldots + \frac{2^{m-1}}{(2^{m-1})^p}$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \ldots + \frac{1}{(2^{(m-1)})^{(p-1)}}$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \frac{1}{2^{3(p-1)}} + \ldots\ldots$$

$$+ \frac{1}{2^{(m-1)(p-1)}}$$

$$= \frac{\left[1 - \frac{1}{(2^{p-1})^m}\right]}{\left(1 - \frac{1}{2^{p-1}}\right)}$$

$$= \frac{1}{1 - \frac{1}{2^{p-1}}}\left[1 - \frac{1}{(2^{p-1})^m}\right]$$

$$\therefore \quad S_{(2^m-1)} < 2\left[1 - \frac{1}{(2^{p-1})^m}\right]$$

$$\therefore \quad S_{(2^m-1)} < 2.$$

Now, for each positive integer n, we can find another positive integer m such that

$$n < 2^m - 1.$$

$\therefore \quad S_n < S_{(2^m-1)} < 2$, as (S_n) is a monotone increasing sequence.

$\therefore$ The monotone increasing sequence (S_n) is bounded above and hence it is convergent.

$\therefore$ The series $\Sigma \frac{1}{n^p}$ converges if $p > 1$.

From the above theorem, it follows immediately that $\Sigma \frac{1}{n^2}$. $\Sigma \frac{1}{n^{3/2}}$, $\Sigma \frac{1}{n^4}$ are all convergent where as $\Sigma \frac{1}{\sqrt{n}}$, $\Sigma \frac{1}{n^{1/3}}$, $\Sigma \frac{1}{n^{1/100}}$ are all divergent.

We can now effectively use the comparison tests and the properties of the standard series $\Sigma \frac{1}{n^p}$ to discuss the convergence of many series. We give below a few illustrative examples.

Illustrative Examples

Ex. 1. Show that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is a convergent series.

Sol. We know that the geometric series

$\Sigma \left(\frac{1}{2}\right)^{n-1}$ is a convergent series.

We have,

$$\frac{1}{1!} \leqslant \frac{1}{2^0},$$

$$\frac{1}{2!} \leqslant \frac{1}{2!},$$

$$\frac{1}{3!} \leqslant \frac{1}{2^2}, \text{ etc.}$$

and in general,

$$\frac{1}{n!} \leqslant \frac{1}{2^{n-1}} \text{ for all } n.$$

$\therefore$ Each term of the series $\Sigma \frac{1}{n!}$ is less than or equal to the corresponding term of the series $\Sigma \left(\frac{1}{2}\right)^{n-1}$ which is known to be convergent.

$\therefore$ By the comparison test (1), the given series is convergent.

Ex. 2. Examine the following series for convergence.

(*ii*) $\sum \frac{n^2}{n^4+2}$; (*ii*) $\sum \frac{n}{n^2+1}$; (*iii*) $\sum \frac{\sqrt{n}+1}{\sqrt{n^3+n^2+1}}$

Sol. (*i*) Consider the series $\sum \frac{n^2}{n^4+2}$.

We have, $\frac{n^2}{n^4+2} < \frac{n^2}{n^4} = \frac{1}{n^2}$ for all n.

$\therefore$ $\frac{n^2}{n^4+2} < \frac{1}{n^2}$.

But $\sum \frac{1}{n^2}$ is known to be a convergent series.

By the comparison test, the series

$$\sum \frac{n^2}{n^4+2} \text{ is convergent.}$$

(*ii*) For the series $\sum \frac{n}{n^2+1}$ we have

$$\frac{n}{n^2+1} \geqslant \frac{n}{n^2+n^2} = \frac{1}{2n} \text{ for all } n.$$

$$\therefore \frac{n}{n^2+1} \geqslant \frac{1}{2n} \text{ for all } n.$$

However, the series $\sum \frac{1}{n}$ and $\sum \frac{1}{2n}$ are known to be divergent.

By the comparison test (2), the series $\sum \frac{n}{n^2+1}$ is divergent.

(*iii*) Now consider the series $\sum \frac{\sqrt{n}+1}{\sqrt{n^3+n^2+1}}$.

We have,

$$\frac{\sqrt{n}+1}{\sqrt{n^3+n^2+1}} \leqslant \frac{\sqrt{n}+\sqrt{n}}{\sqrt{n^3}} = \frac{2\sqrt{n}}{\sqrt{n^3}} = \frac{2}{n^{3/2}},$$

for all n.

$$\therefore \frac{\sqrt{n}+1}{\sqrt{n^3+n^2+1}} \leqslant \frac{2}{n^{3/2}} \text{ for all } n.$$

However, the series $\sum \frac{2}{n^{3/2}}$ is known to be convergent.

$\therefore$ By the comparison test (1), the given series is convergent.

Remark. In the solutions of above examples, we have skillfully employed the inequalities. However, it is not always possible to obtain simplified expressions through inequalities to discuss the convergence of the given series. For this purpose, we have tests known as *ratio tests* which are much easier to handle.

Ratio Test

(*i*) Simple Ratio Test

Let Σa_n and Σb_n be series of positive terms and further suppose that the convergence property of the series Σb_n is already known.

If $\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right)$ exists and it is unique and finite number, then

both Σa_n and Σb_n converge together or both Σa_n and Σb_n diverge together.

i.e. If Σb_n is convergent then so is Σa_n and if Σb_n is divergent then so is Σa_n.

(*ii*) **D'Alembert's Ratio Test**

Let Σa_n be a series of positive terms and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and is equal to l.

If $l < 1$, then the series is convergent, $l > 1$, then the series is divergent. The test is *in-conclusive* if $l = 1$.

Both these tests are extremely useful though each one of them has some short-coming. The following illustrative plesexam demonstrate how these tests are to be employed to determine ths convergence of the given infinite series.

Ex. 1. Examine the convergence of the following infinite series.

(*i*) $\sum \frac{1}{\sqrt{n+\sqrt{n+1}}}$; (*ii*) $\sum \frac{n+1}{n^3+2n^2+5}$

(*iii*) $\sum \frac{7n+6}{n^2\sqrt{n}+5}$

Sol. (*i*) Consider the series $\sum \frac{1}{\sqrt{n+\sqrt{n+1}}}$. For the purpose of ratio test, choose the series $\sum \frac{1}{n^{1/2}}$.

Now, the nth term of the given series is $\frac{1}{\sqrt{n+\sqrt{n+1}}}$ where as the nth term of the chosen series is $\frac{1}{\sqrt{n}}$.

$$\therefore \lim_{n\to\infty} \left[\frac{\left(\frac{1}{\sqrt{n+\sqrt{n+1}}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} \right] = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{\sqrt{n+\sqrt{n+1}}} \right)$$

$$= \tfrac{1}{2}$$

$\therefore$ Both the series converge together or diverge together.

But $\sum \frac{1}{n^{1/2}}$ is known to be divergent.

$\therefore$ The given series is divergent .

(*ii*) $\sum \frac{n+1}{n^3+2n^2+5}$.

For the purpose of ratio test, choose the series $\sum \frac{1}{n^2}$

$$\therefore \lim_{n\to\infty} \left[\frac{\left(\frac{n+1}{n^3+2n^2+5}\right)}{\left(\frac{1}{n^2}\right)}\right] = \lim_{n\to\infty} \left[\frac{n^2(n+1)}{n^3+2n^2+5}\right]$$

$$= 1.$$

$\therefore$ Both the series converge together or diverge together.

But $\sum \frac{1}{n^2}$ is known to be convergent.

$\therefore$ The given series $\sum \frac{n+1}{n^3+2n^2+5}$ is convergent.

(*iii*) $\sum \frac{7n+6}{n^2\sqrt{n}+5}$

For the purpose of ratio test, choose the series $\sum \frac{1}{n\sqrt{n}}$ or $\sum \frac{1}{n^{3/2}}$.

Now, we have

$$\lim_{n\to\infty} \left[\frac{\left(\frac{7n+6}{n^2\sqrt{n}+5}\right)}{\left(\frac{1}{n\sqrt{n}}\right)}\right] = \lim_{n\to\infty} \left[\frac{(7n+6)(n\sqrt{n})}{n^2\sqrt{n}+5}\right]$$

$$= 7$$

$\therefore$ Both the series converge together or diverge together.

But $\sum \frac{1}{n\sqrt{n}}$ is known to be convergent.

$\therefore$ The given series $\sum \frac{7n+6}{n^2\sqrt{n}+5}$ is convergent.

Remark. To choose a suitable series for the purpose of ratio test, the following procedure may be adopted. If Σa_n is the given series, then let p be the number obtained by subtracting the highest index of n in the numerator of a_n from the highest index of n in the denominator of a_n. Then, $\sum \frac{1}{n^p}$ is the appropriate series which can be used in the ratio test.

For example, in the series $\sum \frac{7n+6}{n^2\sqrt{n}+5}$, the highest index of n in the numerator of $\frac{7n+6}{n^2\sqrt{n}+5}$ is 1 whereas the highest index

of n in the denominator of the same term is $\frac{5}{2}$. Clearly $p = \frac{5}{2} - 1 = \frac{3}{2}$.

$\sum \frac{1}{n^{3/2}}$ is the appropriate series which can be used in the ratio test. Observe that in Ex. 1 (*iii*), we did exactly the same.

Ex. 2. Discuss the convergence of the following infinite series,

(*i*) $\sum \frac{n!}{2^n}$; (*ii*) $\sum \frac{n}{3^{+n}}$.

Sol. (*i*) Let $\sum a_n$ be the given infinite series.

Then we have,

$$a_n = \frac{n!}{2^n} \text{ and}$$

$$a_{n+1} = \frac{(n+1)!}{2^{n+1}} .$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{2^{n+1}} \times \frac{2^n}{n!}$$

$$= \frac{(n+1)!}{n!} \times \frac{2^n}{2^{n+1}}$$

$$= \frac{n+1}{2}$$

$$\therefore \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = \infty .$$

$$\therefore \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) > 1.$$

$\therefore$ By D'Alembert's ratio test, the series is divergent.

(*ii*) Let $\sum a_n$ be the given infinite series.

Then we have,

$$a_n = \frac{n}{3^n} \text{ and}$$

$$a_{n+1} = \frac{n+1}{3^{n+1}}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{n+1}{3^{n+1}} \times \frac{3^n}{n}$$

$$= \frac{n+1}{n} \times \frac{1}{3} .$$

$$\therefore \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = \frac{1}{3} < 1.$$

$\therefore$ By D' Alembert's ratio test, the given series is convergent.

Ex. 3. Show that the following infinite series is convergent.

$$\sum \frac{x^n}{n!} \; ; \; x > 0.$$

Sol. Let $\sum a_n$ be the given infinite series.

Then we have,

$$a_n = \frac{x^n}{n!} \text{ and}$$

$$a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\therefore \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left[\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} \right]$$

$$= \lim_{n \to \infty} \left[\frac{x^{n+1}}{x^n} \times \frac{n!}{(n+1)!} \right]$$

$$= \lim_{n \to \infty} \left[\frac{x}{n+1} \right] = 0 \text{ for all } x > 0.$$

$$\therefore \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = 0 < 1.$$

$\therefore$ By D' Alembert's ratio test, the given infinite series is convergent.

Ex. 4. Examine the following series for convergence.

$$(i) \sum n^p x^n \; ; \; x > 0 \; ; \quad (ii) \sum 5^n x^n \; ; \; x > 0.$$

Sol. Let $\sum a_n$ be the given series.

Then we have,

$$a_n = n^p \, x^n \text{ and}$$

$$a_{n+1} = (n+1)^p \, x^{n+1}$$

$$\therefore \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left[\frac{(n+1)^p \, x^{n+1}}{n^p \, x^n} \right]$$

$$= \lim_{n \to \infty} \left[\left(1 + \frac{1}{n} \right)^p x \right] = x$$

$$\therefore \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = x.$$

$\therefore$ The given series is convergent if $0 < x < 1$ and divergent if $x > 1$.

However, D'Alembert's test is inconclusive if $x = 1$.

Now, suppose that $x = 1$.

Then the infinite series becomes $\sum n^p$.

Clearly it is convergent if $p < -1$ and divergent if $p \geqslant 1$.

$\therefore$ The infinite series $\sum n^p x^n$ is convergent for all values of p and $0 < x < 1$.

It is also convergent when $p < 1$ and $x = 1$.

The infinite series is divergent for all values of p and $x > 1$.

It is also divergent when $p \geqslant 1$ and $x = 1$.

(ii) Let $\sum b_n$ be the given infinite series.

Then clearly,

$$b_n = 5^n x^n \text{ and}$$

$$b_{n+1} = 5^{n+1} x^{n+1}.$$

$$\therefore \lim_{n \to \infty} \left(\frac{b_{n+1}}{b_n} \right) = \lim_{n \to \infty} \left(\frac{5^{n+1} x^{n+1}}{5^n x^n} \right)$$

$$= 5x.$$

$\therefore$ The infinite series is convergent if $5x < 1$ and divergent if $5x > 1$. The D'Alembert's test fails if $x = \frac{1}{5}$.

Put $x = \frac{1}{5}$ in the given series. Then every term of the series is 1. Clearly $S_n = n$ and hence the series is divergent.

$\therefore$ The given series is divergent if $x \geqslant \frac{1}{5}$ and convergent if $0 < x < \frac{1}{5}$.

Remark 1. In the D'Alembert's ratio test for a series Σa_n, if $\lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = 1$, then the test is inconclusive. The following examples illustrate this further.

For the series $\sum \frac{1}{n}$, we have

$\lim_{n \to \infty} \left(\frac{(a_{n+1})}{a_n} \right) = \lim_{x \to \infty} \left(\frac{n+1}{n} \right) = 1$, but we know that the series $\sum \frac{1}{n}$ is divergent.

In case of the series $\sum \frac{1}{n^2}$ we have $\lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left[\frac{(n+1)^2}{n^2} \right] = 1$, but we know that the series $\sum \frac{1}{n^2}$ is convergent.

This implies that for the convergent as well as divergent series, $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$ can be 1.

Remark 2. Let Σa_n be a series of positive terms and let S_n be its partial sum.

Clearly $S_n = a_1 + a_2 + \ldots\ldots + a_n$ and

$$S_{n+1} = a_1 + a_2 + \ldots\ldots + a_n + a_{n+1}.$$

$\therefore$ $S_{n+1} - S_n = a_{n+1} > 0$

$\therefore$ (S_n) is monotone increasing.

$\therefore$ The sequence (S_n) cannot oscillate.

It means that either the sequence (S_n) is convergent or it is divergent.

It can be easily shown that if the series Σa_n is convergent, then

$$\lim_{n\to\infty} a_n = 0. \qquad i.e.\ a_n \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

We can use this result skillfully to discuss the convergence properties of the infinite series Σa_n.

Let Σa_n be a series of positive terms. Then, as discussed above, it cannot oscillate. If it is known that $\lim_{n\to\infty} a_n \neq 0$ then the series is not convergent. Therefore the series Σa_n must be divergent. The following examples illustrate this point further.

Ex. 5. Show that the following series are divergent without using any ratio tests.

$$(i)\ \sum \frac{n+1}{2n+5} \qquad (ii)\ \sum \frac{n^2+2n+3}{4n^2+8n+9}$$

Sol. (*i*) The nth term of the series $\sum \frac{n+1}{2n+5}$ is $\left(\frac{n+1}{2n+5}\right)$.

Now, $\lim_{n\to\infty}\left(\frac{n+1}{2n+5}\right) = \frac{1}{2}$.

Since $\sum \frac{n+1}{2n+5}$ is a series of positive terms, it *cannot* oscillate. Also, the nth term of the series does not tend to zero and therefore it cannot converge. Hence, the given series is divergent.

(*ii*) The series $\sum \frac{n^2+2n+3}{4n^2+8n+9}$ is a series of positive terms and therefore it can oscillate.

Now, $\lim_{n\to\infty}\left(\frac{n^2+2n+3}{4n^2+8n+9}\right) = \frac{1}{4}$.

$\therefore$ The series cannot converge and therefore it must diverge.

EXERCISE 1·8

1. Examine the following series for convergence.

(*i*) $\sum \frac{1}{n+x}$; (*ii*) $\sum \frac{1}{n^2+x}$; (*iii*) $\sum \frac{1}{n^p+x}$;

(*iv*) $\sum \frac{1}{n^2+n+1}$; (*v*) $\sum \frac{n+1}{n^2+2}$; (*vi*) $\sum \frac{2^n-1}{5^n+1}$;

(*vii*) $\sum \frac{n+1}{n^3+3n^2}$; (*viii*) $\sum \frac{5^n+1}{3^n-2}$

(*ix*) $\sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$ (*x*) $\sum \frac{\sqrt{\quad}-\sqrt{n-1}}{n}$;

(*xi*) $\sum \left(\frac{n}{1+2n^3}\right)^{1/2}$ (*xii*) $\sum \frac{|\sin n\theta|}{n^2+1}$;

(*xiii*) $\sum x^n |\cos(\theta/n)|$; (*xiv*) $\sum \left(\sqrt{2n+3}-\sqrt{2n+1}\right)$

(*xv*) $\sum \frac{|\sin(\theta/n)|}{n^p}$

2. Examine the following series for convergence (Take $x>0$).

(*i*) $x+\frac{x^3}{3!}+\frac{x^5}{5!}$ .....+ ..... $+\frac{x^{2n-1}}{(2n-1)!}+$.....

(*ii*) $1+\frac{x^2}{2!}+\frac{x^2}{4!}+$ ..... +..... $+\frac{x^{2n}}{(2n)!}+$.....

(*iii*) $x+\frac{x^2}{2}+\frac{x^3}{3}+$ ..... + ... $+\frac{x^n}{n}+$ .....

(*iv*) $\sum \frac{1}{(n+1)(2n+3)}$; (*v*) $\sum \frac{\sqrt{n+1}}{2n-1}x^n$;

(*vi*) $\sum 2^n x^n$ (*vii*) $\sum \frac{x^n}{n^p}$; $p>0$;

(*viii*) $\sum n^n x^n$; (*ix*) $\sum \frac{x^n n^p}{n!}$; $p>0$

(*x*) $\sum \frac{1}{n \cdot 2^n}$

3. Show that in the case of the following series the *D'* Alemberts ratio test fails but the simple ratio test is decisive.

(*i*) $\sum \frac{1}{n+3}$; (*ii*) $\sum \frac{n+1}{n^3+5n+6}$;

(*iii*) $\sum \frac{n^2+2n+3}{n^4-2n^2+5}$

4. Show that for the series given below, it is simpler to apply the ratio tests than the comparison tests.

(*i*) $\sum \frac{3^n+1}{5^n-1}$; (*ii*) $\sum \frac{|x^n}{2^n+1}$.

5. Show that the following infinite series are divergent.

$(i)\ \sum \frac{2n}{n^2+2}$; $(ii)\ \sum \frac{2+5(n-1)}{3+7(n)}$;

$(iii)\ \sum n\left|\text{Sin}\left(\frac{\pi}{4n}\right)\right|$; $(iv)\ \sum \frac{3n+2^{1/n}}{6n+5^{1/n}}$.

ANSWERS

1. (i) Divergent ; (ii) Convergent ; (iii) Convergent for $p>1$ and divergent for $p \leqslant 1$; (iv) Convergent ; (v) divergent ; (vi) Convergent ; (vii) Convergent ; $(viii)$ divergent ; (ix) divergent ; (x) Convergent ; (xi) divergent ; (xii) Convergent ; $(xiii)$ Convergent if $0<x<1$; divergent if $x \geqslant 1$; (xiv) divergent ; (xv) Convergent if $p>2$.

2. (i) Convergent ; (ii) Convergent ; (iii) Convergent ; for $x<1$ Other wise divergent ; (iv) Convergent ; (v) Convergent if $x<1$, Otherwise divergent ; (vi) Convergent if $0 < x < \frac{1}{2}$; (vii) Covergent for $0 < x<1$ and all p ; $(viii)$ divergent ; (ix) Convergent (x) Convergent.

2

Limits and Continuity

2·1 Intervals

By (a, b) we shall denote the set of real numbers,

$$\{x/x \in, \mathrm{R}, a < x < b\}$$

The real numbers a and b are called the end points of the interval. In this case they do not belong to the set. The set is then called an *open interval.* When a and b belong to the set, *i.e.* the set of real numbers,

$$\{x/x \in \mathrm{R},\ a \leqslant x \leqslant b\}$$

is called a *closed interval.* This is denoted by $[a, b]$. $(b-a)$ is called the length of both the intervals (a, b) and $[a, b]$.

Similarly $[a, b)$ and $(a, b]$ are *half-open* or *semi-open* intervals. Thus,

$$[a, b) = \{x/x \in \mathrm{R},\ a \leqslant x < b\}$$
$$\text{and } (a, b] = \{x/x \in \mathrm{R},\ a < x \leqslant b\}.$$

Note. Do not confuse the open interval (a, b) with the point with co-ordinates (a, b). The context will always make it clear whether (a, b) is an interval or the coordinates of a point.

Illustrative examples

Ex. 1. Represent the following sets of real numbers as inter-. Represent them also by diagrams.

(i) $\{x/x \in \mathrm{R}, -2 < x < 3\}$ (ii) $\{x/x \in \mathrm{R}, 3 \leqslant x \leqslant 5\}$

(iii) $\{x/x \in \mathrm{R},\ 1 \leqslant x < 4\}$ (iv) $\{x/x \in \mathrm{R}, -4 < x \leqslant -1\}$

Sol. (i) $\{x/x \in \mathrm{R}, -2 < x < 3\}$ is the open interval $(-2, 3)$

(ii) $\{x/x \in \mathrm{R},\ 3 \leqslant x \leqslant 5\}$ is the closed interval $[3, 5]$

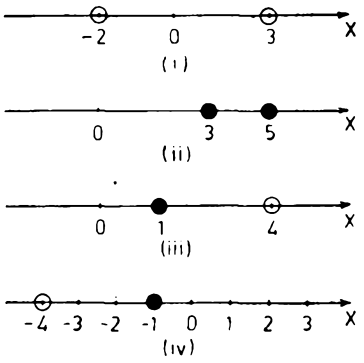

Fig. 2·1.

(*iii*) $\{x/x \in \mathbb{R},\ 1 \leqslant x < 4\}$ is the half-open interval $[1, 4)$

(*iv*) $\{x/x \in \mathbb{R}, -4 < x \leqslant -1\}$ is the half open interval $(-4, -1]$

Ex. 2. Represent the following intervals in set notation.

(*i*) $(2, 5)$, (*ii*) $[-1, 3]$, (*iii*) $[0, 1)$, (*iv*) $(-1, 1]$

Sol.

(*i*) $(2, 5) = \{x/x \in \mathbb{R},\ 2 < x < 5\}$

(*ii*) $[-1, 3] = \{x/x \in \mathbb{R},\ -1 \leqslant x \leqslant 3\}$

(*iii*) $[0, 1) = \{x/x \in ,\ \mathbb{R}0 \leqslant x < 1\}$

(*iv*) $(-1, 1] = \{x/x \in \mathbb{R},\ -1 < x \leqslant 1\}$

2·2. Neighbourhoods

You have learnt earlier that, the modulus of x or the absolute value of x is,

$$|x| = -x \quad \text{if } x \leqslant 0$$
$$= x \quad \text{if } x > 0$$

You also know that $|x|$, geometrically is the distance of x from the origin. Let us consider a few examples.

Ex. 1. $|x| = 2$. This means, $x = 2$ or $x = -2$

Ex. 2. $|x| < 2$. This means

$-x < 2$ for $x \leqslant 0$ *i.e.* $x > -2$

and $x < 2$ for $x > 0$

$\therefore$ if $|x| < 2$, then $-2 < x < 2$.

This is also evident from the geometrical interpretation of $|x|$,

$|x| < 2$, means, the distance of x from the origin 0 is less than 2.

$\therefore$ x must lie between -2 and 2,

(points A and B in the figure.)

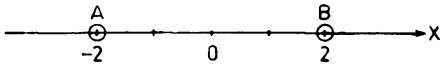

Fig. 2·2.

$\therefore$ The solution set of $|x| < 2$ is,

$$\{x \mid x \in \mathbb{R}, -2 < x < 2\}$$

i.e. the open interval $(-2, 2)$.

Ex. 3. $|x| \leqslant 2$

As in ex. 2, the solution set in this case is,

$$\{x \mid x \in \mathbb{R}, -2 \leqslant x \leqslant 2\}$$

i.e. the closed interval $[-2, 2]$

You also know that,

$$|x-a| = a-x \text{ for } x \leqslant a$$
$$= x-a \text{ for } x > a$$

Further, geometrically, $|x-a|$ represents the distance of x from a. Let us consider a few examples.

Ex. 1. $|x-5| = 2$

$\therefore$ $5-x = 2$ for $x \leqslant 2$

or $x-5 = 2$ for $x > 2$

$\therefore$ $x = 3$ or $x = 7$

This is also evident from the geometrical meaning of $|x-5|$.

$|x-5| = 2$ means, the distance of 'x' from '5' is 2

$\therefore$ $x = 3$ or $x = 7$

Ex. 2. $|x-5| < 2$

$\therefore$ $5-x < 2$ for $x \leqslant 5$

$x-5 < 2$ for $x > 5$

$\therefore$ $3 < x$ and $x < 7$ *i.e.* $3 < x < 7$

This is also evident from the geometrical meaning of $|x-5|$.

$|x-5|$ = Distance of x from '5'

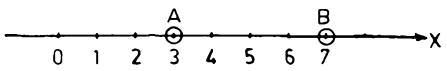

Fig. 2·3.

$\therefore$ $|x-5| < 2$ means that the distance of 'x' from '5' is less than 2. As can be seen from the figure, this requires that x lies between 3 and 7. (Points A and B in the figure.)

$\therefore$ The solution set of $|x-5|<2$ is,

$\{x/x \in \mathbb{R},\ 3 < x < 7\}$

i.e. the open interval (3, 7).

Ex. 3. $|x-5| \leqslant 2$

As in ex. 2 above you can verify that the solution set of this inequality is,

$\{x/x \in \mathbb{R},\ 3 \leqslant x \leqslant 7\}$ *i.e.* the closed interval [3, 7]

In general the solution set of the inequality, $|x-a| < \delta$, is

$\{x/x \in \mathbb{R},\ a-\delta < x < a+\delta\}$

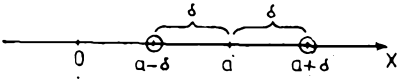

Fig. 2·4.

i.e. the open interval $(a-\delta, a+\delta)$

Also taking $a = 0$, the solution set of $|x| < \delta$ is the open interval $(-\delta, \delta)$.

Observe that the solution sets of,

$$|x| < \delta \text{ and } |x-a| < \delta$$

are both open intervals *e.g.* $(-\delta, \delta)$ and $(a-\delta, a+\delta)$.

Further, they are intervals of a special kind. In the first, 0 is the midpoint of the interval and its length is 2δ. In the second, the midpoint is a and the length is 2δ. We give special name to such intervals centered around a given point. They are called the *δ-neighbourhoods* of these points. Thus, the *δ-neighbourhood* of a is the open interval,

$$(a-\delta, a+\delta) \text{ } i.e. \text{ } a-\delta < x < a+\delta.$$

For short *δ-neighbourhood* will be written as *δ-nbd.*

Illustrative examples

Ex. 1. Write down the following neighbourhoods as sets and intervals.

(*i*) δ—nbd of 2, (*ii*) 0·01—nbd of 3, (*iii*) ·5—nbd of (—4)

Sol. (*i*) δ—nbd of $2 = 2-\delta < x < 2+\delta$ or $|x-2| < \delta$

(*ii*) ·01—nbd of $3 = 3-{\cdot}01 < x < 3+{\cdot}01$

i.e. $2{\cdot}99 < x < 3{\cdot}01$ or $|x-3| < 0{\cdot}01$

(*iii*) 5-nbd of $(-4) = -4-{\cdot}5 < x < -4+{\cdot}5$

i.e. $-4{\cdot}5 < x < -3{\cdot}5$ or $|x+4| < {\cdot}5$

Ex. 2. Write down the following sets as neighbourhoods.

(*i*) $4{\cdot}9 < x < 5{\cdot}1$

(*ii*) $| x-10 | < {\cdot}25$.

(*iii*) $-3 < x < -1$

Sol. (*i*) $4{\cdot}9 < x < 5{\cdot}1$ is $0{\cdot}1$—nbd of 5.

(*ii*) $| x-10 | < .25$ is $0{\cdot}25$—nbd of 10.

(*iii*) $-3 < x < -1$ is $1{\cdot}0$—nbd of -2.

Deleted neighbourhoods.

In some situations it becomes necessary to consider δ-nbd of a from which a has been deleted *i.e.* omitted. Such a neighbourhood is then called the *deleted* δ*-nbd of* a.

The deleted δ-nbd of a is,

$$0 < | x-a | < \delta \text{ or } a-\delta < x < a+\delta \text{ and } x \neq a.$$

Thus,

(*i*) $0 < | x-2 | < {\cdot}01$ is the deleted $0{\cdot}01$—nbd of 2 and it represents the set of real numbers x lying between $1{\cdot}99$ and $2{\cdot}01$ and excluding the number 2.

(*ii*) $0 < | x+4 | < 2$ is the deleted $2{\cdot}0$—nbd of -4 representing the set of real numbers $-6 < x < -2$ and $x \neq -4$

Examples

Ex. 1. Represent the following sets of real numbers as intervals.

(*i*) $\{x/x \in \mathrm{R}, \quad 0 < x < 2\}$, (*ii*) $\{x/x \in \mathrm{R}, -1 \leqslant x < 1\}$

(*iii*) $\{x/x \in \mathrm{R}, -1 < x \leqslant 2\}$, (*iv*) $\{x/x \in \mathrm{R}, -2 \leqslant x \leqslant 2\}$

Ex. 2. Represent the following intervals as sets

(*i*) $(-1,3)$; (*ii*) $[2, 5]$; (*iii*) $[0, 1]$; (*iv*) $[-3, 0)$; (*v*) $(-2, 3)$; (*vi*) $(0, 1)$.

Ex. 3. Write down the following neighbourhoods as sets and intervals

(*i*) $0{\cdot}1$—nbd of 1 (*ii*) $0{\cdot}01$—nbd of (-1)

(*iii*) $00{\cdot}1$—nbd of 3 (*iv*) $0{\cdot}01$—nbd (deleted) of 4

(*v*) $0{\cdot}2$—deleted nbd. of 0.

(*vi*) $0{\cdot}5$—deleted nbd. of $0{\cdot}3$.

(*vi*) $0{\cdot}3$—deleted nbd. of 2.

Ex. 4. Write down the following sets as neighbourhoods:

(*i*) $-0{\cdot}1 < x < 0{\cdot}1$; (*ii*) $2{\cdot}5 < x < 3{\cdot}5$; (*iii*) $-3{\cdot}9 < x < 4{\cdot}1$; (*iv*) $| x-5 | < 0{\cdot}2$; (*v*) $| x+2 | < 0{\cdot}02$

(*vi*) $0 < |x-6| < 0{\cdot}2$; (*vii*) $0 < |x+5| < 0{\cdot}3$;

(*viii*) $0 < |x| < 0{\cdot}3$; (*ix*) $-3{\cdot}8 < x < 4{\cdot}2, x \neq 4$

(*x*) $4{\cdot}7 < x < 5{\cdot}3, x \neq 5$.

Ex. 5. (*i*) Find two values of x, one greater than 3 and the other less than 3 such that $|x-3| < 0{\cdot}3$.

(*ii*) Find two values of x one less than 5 and the other greater than 5 such that $0 < |x-5| < 0{\cdot}01$. Does $x = 5$ satisfy this inequality ?

(*iii*) Find two values of x, one less than -4 and the other greater than -4 such that $0 < |x+4| < 0{\cdot}2$. Does $x = -4$ satisfy this inequality ?

ANSWERS

1. (*i*) (0,2); (*ii*) $[-1, 1)$; (*iii*) $(-1, 2]$; (*iv*) $[-2, 2]$.
2. (*i*) $\{x \mid x \in \mathbf{R} - 1 < x < 3\}$, (*ii*) $\{x \mid x \in \mathbf{R}, 2 \leqslant x \leqslant 5\}$.
   (*iii*) $\{x \mid x \leqslant \mathbf{R}, 0 \leqslant x \leqslant 1\}$; (*iv*) $\{x \mid x \leqslant \mathbf{R} - 3 \leqslant x < 0\}$.
   (*v*) $\{x \mid \leqslant \mathbf{R}, -2 < x \geqslant 3\}$; (*vi*) $\{x \mid x \leqslant R, 0 < x < 1\}$.
3. (*i*) $0{\cdot}9 < x < 1{\cdot}1$; (*ii*) $-1{\cdot}01 < x < -0{\cdot}9$.
   (*iii*) $2{\cdot}99 < x < 3{\cdot}01$; (*iv*) $3{\cdot}99 < x < 4{\cdot}01, x \neq 4$.
   (*v*) $0{\cdot}2\, x < x < 0{\cdot}2, x \neq 0$; (*vi*) $0 < |x+3| < 0{\cdot}5$.
   (*vii*) $0 < |x-2| < 0{\cdot}3$.
4. (*i*) 0·1 nbd. of 0; (*ii*) 0·5 nbd of 3; (*iii*) 0·1 nod. of 4;
   (*iv*) 0·2 nbd. of 5; (*v*) 0·02 nbd. of -2.
   (*vi*) 0·2 nbd of 6 (deleted); (*vii*) 0·3 deleted nbd. of -5.
   (*viii*) 0·3 deleted nbd. of 0 ; (*ix*) 0·3 deleted nbd. of 4;
   (*x*) 0·3 deleted nbd. of 5;
5. (*i*) $x = 2{\cdot}8, 3{\cdot}1$; (*ii*) 4·999, 5·001, No.

2·3. Behaviour of a function at a point and in its neighbourhood

Consider the functions :

(*i*) $f(x) = \dfrac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(*ii*) $g(x) = x^2$.

We shall examine their behaviour in the neighbourhood of O. For this purpose we construct the following table giving their values at some points in the 0·1 nbd. of 0.

Their graphs are also shown :

x	$f(x)$	$g(x)$
−0·1	−10	0·01
−0·01	−100	0·0001
−0·001	−1000	0·000001
0·0	0	0·0
0·001	1000	0·000001
0·01	100	0·0001
0·1	10	0·01

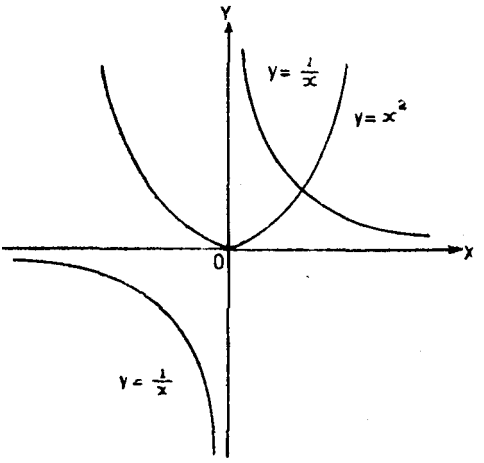

Fig. 2·5.

Observe that,

(*i*) $f(x)$ is negative on the l·h.s. of 0 and positive on the r.h.s. of 0. $g(x)$ is positive on both sides.

(*ii*) $f(0) = 0$ and there is a sudden jump in the values of f as we go from left to right through 0. From very large negative values we go to very large +ve values.

There is no such sudden change in the values of g. This is clearly shown in the graph. The graph of f is in two distinct parts. For $x < 0$ in the 3rd quadrant and for $x > 0$ in the 1st quadrant. On the other hand, the graph of g is a continuous curve passing through the origin.

(*iii*) The values of f are decreasing as we approach 0 from the left and are increasing as we approach 0 from the right. The values of g go on decreasing as we approach 0 from either side.

(*iv*) $f(0) = 0$ and $g(0) = 0$. Note that f is defined differently for $x \neq 0$ and $x = 0$.

The student can study other functions similarly in the neighbourhood of some selected values of the argument.

From such examples we see that,

(*i*) A function may behave differently on the two sides of a point, in its neighbourhood.

(*ii*) A function may suddenly change as x passes through a particular value.

(*iii*) A function may not even exist at a point but may be well defined in its neighbourhood.

In general, we may say that the behaviour of a function at a point and in its neighbourhood may be quite different. The student should bear this in mind when we discuss the concepts of limit and continuity in the remaining part of the chapter.

2·4. Limit of a function at a point.

You have been introduced to the concept of the limit of a function in an elementary way in the XIth standard. We shall now make the concept more precise. The following explanation of the concept of the limit of a function f as x tends to a was given.

"If $|f(x)-l|$ can be made less than any given positive number for values of x sufficiently close and closer to a but not equal to a, then we say that $f(x)$ tends to the limit as x tends to l a."

We shall now put the idea in the above explanation in mathematical language. Let us consider a few examples.

Ex. 1. $f(x) = 2x+3$

We consider the limit of the function as x tends to 1. A little calculation will convince you that $f(x)$ approaches 5 as x tends to 1.

Now,

$$|f(x)-5| = |2x+3-5|$$
$$= 2|x-1|$$

Suppose we are given any positive number, for example, 0·02 Then,

$|f(x)-5| < 0{\cdot}02$ if $2|x-1| < 0{\cdot}02$

$\therefore |f(x)-5| < 0{\cdot}02$ if $|x-1| < 0{\cdot}01$.

Similarly,

$|f(x)-5| < 0{\cdot}01$ if $|x-1| < 0{\cdot}005$.

In general, if $\varepsilon > 0$ is given,

$$| f(x)-5 | < \varepsilon \text{ if } | x-1 | < \frac{\varepsilon}{2} .$$

[This means that if $x \in \frac{\varepsilon}{2}$ — nbd of 1 then.

$f(x)$ ε—nbd of 5]-

As stated in the explanation quoted above we observe that $| f(x)-5 | <$ any given positive number ε for values of as close and closer to 1 than $1-\frac{\varepsilon}{2}$ and $1+\frac{\varepsilon}{2}$. The above explanation may now be stated with greater mathematical precision as :

Given $\varepsilon > 0$ we can find $\delta > 0 \left(= \frac{\varepsilon}{2} \right)$ such that,

$$| f(x)-5 | < \varepsilon \text{ for all } | x-1 | < \delta = \frac{\varepsilon}{2} .$$

i.e., for all $1-\frac{\varepsilon}{2} < x < 1+\frac{\varepsilon}{2}$.

We have seen that a function may behave quite differently in the neighbourhood of a and at $x = a$. Hence, it is mentioned in the explanation that, 'for values of x close and closer to a but not equal to a'. We may therefore write for the above function,

$$| f(x)-5 | < \varepsilon \text{ for all } 0 < | x-1 | < \frac{\varepsilon}{2} .$$

$0 < | x-1 |$ ensures that $x \neq 1$. We have here a deleted neighbourhood of a.

In the case of this function we note that we need not have a deleted neighbourhood of 1 for the first inequality to hold *e.g.*, $| f(x)-5 | < \varepsilon$. The next example brings out the necessity of admitting only deleted neighbourhoods. The importance of admitting only deleted neighbourhoods when considering limits lies in the fact that when we consider the limit as x tends to a, we consider values in the neighbourhood of a but definitely not equal to a. Thus the very concept x tends to a implies a deleted neighbourhood.

Ex. 2. $f(x) = \dfrac{2x^2-x-1}{x-1}$, $x \neq 1$; $f(1) = 0$.

We consider the limit of this function as x tends to 1. The methods learnt in the XIth Standard will enable you to find the limit, which is 3

Now, $| f(1)-3 | = | 0-3 | = 3$

and $| f(x)-3 | = \left| \frac{2x^2-x-1}{x-1} - 3 \right|$ for $x \neq 1$

$$= \left| \frac{(2x+1)(x-1)}{(x-1)} - 3 \right|$$

$$= | (2x+1)-3 | , \quad \because \quad x \neq 1$$

$\therefore \quad | f(x)-3 | = 2 | x-1 |$ for $x \neq 1$.

$= 3$ for $x = 1$.

Suppose we are given a positive number 0·01. Then,

$| f(x)-3 | < 0{\cdot}01$ if $2 | x-1 | < 0{\cdot}01$ and $x \neq 1$

i.e. $| f(x)-3 | < 0{\cdot}01$ if $| x-1 | < 0{\cdot}005$ and $x \neq 1$.

But for $x = 1$, $| f(x)-3 | = 3 \not< 0{\cdot}01$.

Similarly, given any positive number ε,

$| f(x)-3 | < \varepsilon$ if $| x-1 | < \frac{1}{2}\varepsilon$ and $x \neq 1$

but for $x = 1$, $| f(1)-3 | = 3 \not< \varepsilon$

Hence, $| f(x)-3 | < \varepsilon$ for values of x closer to 1 than $1-\frac{\varepsilon}{2}$ and $1+\frac{\varepsilon}{2}$ but not for $x = 1$. For $x = 1$, $| f(1)-3 | = 3 \not< \varepsilon$.

The explanation of the limit of $f(x)$ as x tends to a can now be written in a precise mathematical form :

Def. 1. If given $\varepsilon > 0$, there exists $\delta > 0$ such that

$| f(x)-l | < \varepsilon$ for all $0 < | x-a | < \delta$,

then we say that $f(x)$ tends to the limit l as x tends to a.

This is written as,

$$\lim_{x \to a} (x) = l$$

We shall now apply the definition to one or two functions.

Illustrative Examples

Ex. 1. Prove that if $f(x) = 2x^2-4x+5$, then $\lim_{x \to 1} f(x) = 3$.

Find $\delta > 0$ such that,

$| f(x)-3 | < 0{\cdot}02$ for all $0 < | x-1 | < \delta$.

Sol. We have.

$| f(x)-3 | = | 2x^2-4x+5-3 |$

$= 2 | (x-1)^2 |$

Let $\varepsilon > 0$ be given.

$\therefore \quad |f(x)-3| < \epsilon$ if $2(x-1)^2 < \epsilon$

i.e., if $|x-1| < \sqrt{\frac{\epsilon}{2}}$

$\therefore \quad |f(x)-3| < \epsilon$ if $0 < |x-1| < \sqrt{\frac{\epsilon}{2}}$

[Actually, $|f(x)-3| < \epsilon$ if $|x-1| < \sqrt{\frac{\epsilon}{2}}$ but to show that the definition can be applied we write $0 < |x-1| < \sqrt{\frac{\epsilon}{2}}$

$\therefore$ according to the definition of the limit of a function, $\lim_{x \to 1} f(x) = 3$.

Now, take $\epsilon = 0{\cdot}02$.

$\therefore \quad |f(x)-3| < 0{\cdot}02$ if $0 < |x-1| < \sqrt{\frac{{\cdot}02}{2}} = 0{\cdot}1$

$\therefore \quad \delta = 0{\cdot}1$

Ex. 2. Use the definition to prove that,

$$\lim_{x \to 2} \frac{x+3}{x+2} = \frac{5}{4}.$$

Sol. We have, $f(x) = \frac{x+3}{x+2}$

$$\therefore \quad \left|f(x) - \frac{5}{4}\right| = \left|\frac{x+3}{x+2} - \frac{5}{4}\right|$$

$$= \left|\frac{4x+12-5x-10}{4(x+2)}\right|$$

$$= \left|\frac{2-x}{4(x+2)}\right|$$

$$\therefore \quad \left|f(x) - \frac{5}{4}\right| = \frac{1}{4}\left|\frac{x-2}{x+2}\right|$$

Now let $\epsilon > 0$ be given

$$\therefore \quad \left|f(x) - \frac{5}{4}\right| < \epsilon \text{ if } \frac{1}{4}\left|\frac{x-2}{x+2}\right| < \epsilon.$$

Observe that in the neighbourhood of 2, $x > 0$ and $|x-2| < |x+2|$.

$$\therefore \quad \frac{1}{4}\left|\frac{x-2}{x+2}\right| < \epsilon \text{ if } \frac{|x-2|}{4} < \epsilon \text{ } i.e., \text{ if } |x-2| < 4\epsilon$$

$$\therefore \quad \left|f(x) - \frac{5}{4}\right| < \epsilon \text{ if } 0 < |x-2| < 4\epsilon$$

Thus, if $\epsilon = 0{\cdot}01$, then

$$\left|f(x) - \frac{5}{4}\right| < 0{\cdot}01 \text{ if } 0 < |x-2| < 0{\cdot}04.$$

$\therefore$ by definition, $\lim_{x \to 2} f(x) = \frac{5}{4}$.

2·5. Limit of a function as x tends to ∞

You have seen that in the case of a convergent sequence (t_n), $t_n \longrightarrow l$ as $n \longrightarrow \infty$ in a certain sense. Mathematically this was defined as;

If given $\epsilon > 0$ we can find a positive integer m such that,

$| t_n - l | < \epsilon$ for all $n \geqslant m$,

then we say that, $t_n \longrightarrow l$ as $n \longrightarrow \infty$

Note that, (t_n) is a function whose domain is $\mathbb{N}$. We shall now develop an exactly similar concept for functions whose domain is $\mathbf{R}$, the set of real numbers. Let us first consider some examples·

Ex. 1. $f(x) = \frac{2x+1}{x+1}$, $x \neq -1$.

We prepare the following table.

x:	99	999	9999	99999
$f(x)$	2·01	2·001	2·0001	2·00001

Observe that as we take larger and larger values of x, (*i*) the value of the function approaches closer and closer to 2.

(*ii*) Further

$| f(x) - 2 | = 0{\cdot}00001$ for $x = 99999$, and for $x > 99999$, $| f(x) - 2 | < 0{\cdot}00001$.

This can be proved

We have,

$$| f(x) - 2 | = \left| \frac{2x+1}{x+1} - 2 \right|$$

$$= \left| \frac{2x+1-2x-2}{x+1} \right|$$

$$= \frac{1}{x+1} \text{ if } x > 0.$$

$\therefore$ $| f(x) - 2 | < 0{\cdot}00001$ if $\frac{1}{x+1} < 0{\cdot}00001$

i.e., if $x+1 > 100000$

$\therefore$ $| f(x) - 2 | < 00001$ if $x > 99999$.

In general, if $\epsilon < 0$ is given

$| f(x) - 2 | < \epsilon$ if $\frac{1}{x+1}$

i.e., if $x > \frac{1}{\epsilon} - 1$.

Let $k = \frac{1}{\epsilon} - 1$

$\therefore$ Given $\epsilon > 0$ we can find $k \in \mathbf{R}$ such that,

$|f(x) - 2| < \epsilon$ if $x > k$.

We say in this case that, $f(x) \longrightarrow 2$ as $x \longrightarrow \infty$ and write this as,

$$\lim_{x \to \infty} f(x) = 2.$$

On the basis of the discussion above we now define.

Def. 2· If given $\epsilon > 0$, there exists $k > 0$,

such that,

$|f(x) - l| < \epsilon$ for all $x \geqslant k$,

then we say that $f(x)$ tends to the limit l as x tends to ∞ and write,

$$\lim_{x \to \infty} f(x) = l.$$

Let us now apply this definition to some functions.

Illustrative Example

Ex. If $f(x) = \frac{2x+3}{2x+1}$, use the definition to prove that,

$$\lim_{x \to \infty} f(x) = 1.$$

Sol. Let $\epsilon > 0$ be given. We have

$$|f(x) - 1| = \left| \frac{2x+3}{2x+1} - 1 \right|$$

$$= \left| \frac{2x+3-2x-1}{2x+1} \right|$$

$$= \frac{2}{2x+1} \text{ for } x > 0.$$

$\therefore$ $|f(x) - 1| < \epsilon$ if $\frac{2}{2x+1} < \epsilon$

i.e., $\frac{2x+1}{2} > \frac{1}{\epsilon}$ or $x > \frac{1}{\epsilon} - \frac{1}{2}$.

Write $k = \frac{1}{\epsilon} - \frac{1}{2}$ $\therefore$ $|f(x) - 1| < \epsilon$ if $x > k$

$\therefore$ by definition $\lim_{x \to \infty} f(x) = 1$.

2·6. As is obvious from the above illustration, it will be very inconvenient to use the definition every time.

In order to simplify the process of finding the limits of functions, we depend on some standard limits and the theorems on limits of functions.

Some standard limits

1. $\lim_{x \to a} k = k$ where k is any real number.
2. $\lim_{x \to a} x^n = a^n$ except when $a = 0$ and $n < 0$.
3. $\lim_{x \to \infty} \frac{1}{x^n} = 0$ for all $n > 0$.

Proof. Since $x \longrightarrow \infty$, $x > 0$.

Let $\epsilon > 0$ be given.

$$\therefore \quad \left| \frac{1}{x^n} \right| < \epsilon \text{ if } \frac{1}{x^n} < \epsilon$$

$$\text{i.e., if } x > \left(\frac{1}{\epsilon}\right)^{\frac{1}{n}}$$

Let $k = \left(\frac{1}{\epsilon}\right)^{\frac{1}{n}}$

$$\therefore \quad \left| \frac{1}{x^n} \right| < \epsilon \text{ for all } x > k.$$

$$\therefore \quad \lim_{x \to \infty} \frac{1}{x^n} = 0.$$

4. $\lim_{x \to \infty} \frac{\sin x}{x} = 1$.
5. $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} \cos x = 1$.

Proof. (i) $\lim_{x \to 0} \sin x = \lim_{x \to 0} \frac{\sin x}{x} \times x$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} x$$

$$= 1 \cdot 0 = 0.$$

(ii) $\lim_{x \to 0} \cos x$

We have

$$\lim_{x \to 0} (1 - \cos x) = \lim_{x \to 0} 2 \sin^2 \frac{x}{2}$$

$$= 2 \lim_{x \to 0} \sin \frac{x}{2} \lim_{x \to 0} \sin \frac{x}{2}$$

$$= 2\times 0\times 0 \qquad \because \text{ as } x\to 0,\ \frac{x}{2}\to 0$$

$$\therefore \lim_{x\to 0}(1-\cos x) = 0$$

$$\therefore \lim_{x\to 0} 1 - \lim_{x\to 0}\cos x = 0$$

$$\therefore 1-\lim_{x\to 0}\cos x = 0$$

$$\therefore \lim_{x\to 0}\cos x = 1.$$

6. (*i*) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x} = \log_e a$ (*ii*) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x} = 1$

[We shall assume this result. The proof is beyond the scope of the syllabus of the XIIth standard.]

7. $\lim\limits_{x\to 0} a^x = 1.$

Proof. $\lim\limits_{x\to 0}(a^x-1) = \lim\limits_{x\to 0}\dfrac{a^x-1}{x}\times x$

$$= \lim_{x\to 0}\frac{a^x-1}{x}\lim_{x\to 0} x$$

$$= \log_e a\times 0 = 0.$$

8. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$

Proof. Let n be a positive integer such that,

$$n \leqslant x \leqslant n+1$$

$$\therefore \frac{1}{n+1} \leqslant \frac{1}{x} \leqslant \frac{1}{n}.$$

$$\therefore 1+\frac{1}{n+1} \leqslant 1+\frac{1}{x} \leqslant 1+\frac{1}{n}$$

$$\left(1+\frac{1}{n+1}\right)^n \leqslant \left(1+\frac{1}{x}\right)^x \leqslant \left(1+\frac{1}{n}\right)^n \qquad \ldots(A)$$

Note that, as $x\to\infty$, $n\to\infty$

$\therefore$ Taking limits as $n\to\infty$ in (A).

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n \leqslant \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \leqslant \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Now, $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n+1}\right)^n = \lim\limits_{n\to\infty}\dfrac{\left(1+\dfrac{1}{n+1}\right)^{n+1}}{1+\dfrac{1}{n+1}}$

$$= \frac{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)}$$

$$= \frac{e}{1} = e$$

and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$

$$\therefore\ e \leqslant \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} \leqslant e$$

$$\therefore\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e.$$

⑨ $\lim_{u\to 0}\left(1+u\right)^{\frac{1}{u}} = e$

Proof. We have,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e.$$

Write $u = \frac{1}{x}$ $\quad\therefore$ as $x \to \infty$, $u \to 0$.

$$\therefore\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{u\to 0}\left(1+u\right)^{\frac{1}{u}}$$

$$\therefore\ \lim_{u\to 0}\left(1+u\right)^{\frac{1}{u}} = e$$

2·7. Theorems on limits

The following theorems on limits will be assumed to be true.

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$.

Theorem 1. $\lim_{x\to a}\{f(x)+g(x)\} = \lim_{x\to a} f(x)+\lim_{x\to a} g(x) = l+m.$

Theorem 2. $\lim_{x\to a}\{f(x)-g(x)\} = \lim_{x\to a} f(x)-\lim_{x\to a} g(x) = l-m.$

Theorem 3. $\lim_{x\to a}\{f(x)\cdot g(x)\} = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x) = l.m.$

Theorem 4. $\lim_{x\to a}\{f(x)\div g(x)\} = \lim_{x\to a} f(x) \div \lim_{x\to a} g(x)$

$= l \div m$, if $\lim_{x\to a} g(x) = m \neq 0$

With the help of the standard limits and the above theorems on limits, we can find the limit of a large number of functions.

The simplest case is where $\lim_{x\to a} f(x)$ is $f(a)$ where $f(a)$ exists *i.e.* $f(a)$ is neither infinite, indeterminate or imaginary. This is obtained by just putting a for x in the function $f(x)$.

In case $f(a)$ does not exist, we have to use some simplifications before the theorems on limits and standard forms can be applied to evaluate the necessary limit. We will divide these cases into a number of types and indicate the method of procedure in each case.

Type A. *Algebraic functions.*

Case 1. In this case x tends to a finite number. We simplify the expression giving $f(x)$, by either (*i*) factorisation (*ii*) rationalisation or (*iii*) expansion. This brings out the factor which makes $f(a)$ non-existent. Getting rid of this factor, we then get the limit by using standard forms and theorems.

Illustrative Examples

Case 1 :

Ex. 1. Evaluate the following limits.

(*i*) $\lim_{x\to 2} \dfrac{x^2-5x+6}{(x-2)}$; (*ii*) $\lim_{x\to 8} \dfrac{x^{5/3}-32}{x^{2/3}-4}$;

(*iii*) $\lim_{x\to 1} \dfrac{\sqrt{2x+1}-\sqrt{x+2}}{(x-1)}$.

Sol. (*i*) $\lim_{x\to 2} \dfrac{x^2-5x+6}{(x-2)} = \lim_{x\to 2} \dfrac{(x-2)\,(x-3)}{(x-2)}$

$= \lim_{x\to 2} (x-3)$ $\because\ x\to 2\ \therefore\ x \neq 2$ and $(x-2) \neq 0$

$= 2-3 = -1$

$\therefore\ \lim_{x\to 2} \dfrac{x^2-5x+6}{x-2} = -1$

(*ii*) $\lim_{x\to 8} \dfrac{x^{5/3}-32}{x^{2/3}-4}$

Write $y = x^{1/3}$ $\therefore\ x^{5/3} = y^5$ and $x^{2/3} = y^2$.

Also, as $x\to 8$, $y\to 2$

$\therefore\ \lim_{x\to 8} \dfrac{x^{5/3}-32}{x^{2/3}-4} = \lim_{y\to 2} \dfrac{y^5-2^5}{y^2-2^2}$

$= \lim_{y\to 2} \dfrac{(y-2)(y^4+2y^3+2^2y^2+2^3y+2^4)}{(y-2)\,(y+2)}$

$$= \lim_{y \to 2} \frac{(y^4+2y^3+2^2y^2+2^3y+2^4)}{y+2} = \frac{5 \times 2^4}{4}$$

$$= 20.$$

$$(iii) \lim_{x \to 1} \frac{\sqrt{2x+1}-\sqrt{x+2}}{(x-1)} = \lim_{x \to 1} \frac{\sqrt{2x+1}-\sqrt{x+2}}{x-1} \times \frac{\sqrt{2x+1}+\sqrt{x+2}}{\sqrt{2x+1}+\sqrt{x+2}}$$

$$= \lim_{x \to 1} \frac{2x+1-x-2}{(x-1)(\sqrt{2x+1}+\sqrt{x+2})}$$

$$= \lim_{x \to 1} \frac{x-1}{(x-1)(\sqrt{2x+1}+\sqrt{x+2})}$$

$$= \lim_{x \to 1} \frac{1}{\sqrt{2x+1}+\sqrt{x+2}} \quad \because x \to 1,\ x \neq 1. \text{ and } x-1 \neq 0$$

$$= \frac{1}{\sqrt{2 \cdot 1+1}+\sqrt{1+2}} = \frac{1}{2\sqrt{3}}$$

$$\therefore \lim_{x \to 1} \frac{\sqrt{2x+1}+\sqrt{x+2}}{x-1} = \frac{1}{2\sqrt{3}}$$

Case 2. In evaluating limits of functions of the type $f(x) = \frac{P(x)}{Q(x)}$ as $x \longrightarrow \infty$

where $P(x)$ and $Q(x)$ are polynomials, divide the numerator and the denominator by x^n where n is the degree of $Q(x)$. Then take the limits of the numerator and denominator separately. The limit is finite if $P(x)$ and $Q(x)$ are of the same degree as in (*i*) below. It is 0 if the degree of $Q(x)$ is greater than that of $P(x)$ as in (*ii*) The function tends to ∞ if the degree of $P(x)$ is greater than that of $Q(x)$ as in (*iii*).

Ex. 2. Find the limits of the following functions as x tends to ∞.

$$(i) \frac{3x+2}{5x+1}; \quad (ii) \frac{2x+3}{x^2+2x+5}; \quad (iii) \frac{x^3-x^2+1}{x^3+x+2}.$$

Sol. (*i*) $\lim_{x \to \infty} \frac{3x+2}{5x+1} = \lim_{x \to \infty} \frac{3+\frac{2}{x}}{5+\frac{1}{x}}$ (dividing the numerator and denominator by x).

$$= \frac{\lim_{x \to \infty} \left(3+\frac{2}{x}\right)}{\lim_{x \to \infty} \left(5+\frac{1}{x}\right)}$$

$$= \frac{3+0}{5+0} \qquad \because \lim_{x\to\infty} \frac{1}{x^n} = 0 \text{ if } n>0.$$

$$\therefore \lim_{x\to\infty} \frac{3x+2}{5x+1} = \frac{3}{5}.$$

(*ii*) $$\lim_{x\to\infty} \frac{2x+3}{x^2+2x+5} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{2}{x}+\frac{5}{x^2}}$$ (dividing numerator and denominator by x^2).

$$= \frac{0+0}{1+0+0} = 0.$$

$$\therefore \lim_{x\to\infty} \frac{2x+3}{x^2+2x+5} = 0.$$

(*iii*) $$\lim_{x\to\infty} \frac{x^3-x^2+1}{x^2+x+2} = \lim_{x\to\infty} \frac{x-1+\frac{1}{x^2}}{1+\frac{1}{x}+\frac{2}{x^2}}$$

$$= \frac{\infty-1+0}{1+0+0}$$

$$= \infty$$

Case 3. In this case we simplify the function by rationalisation before applying the procedure of case 2).

Ex. Evaluate the following limits.

(*i*) $$\lim_{x\to\infty} \frac{\sqrt{x^2+5} - \sqrt{x^2+1}}{3x};$$

(*ii*) $$\lim_{x\to\infty} \sqrt{x+1}\left(\sqrt{x+5} - \sqrt{x+4}\right)$$

Sol. (*i*) $$\lim_{x\to\infty} \frac{\sqrt{x^2+5} - \sqrt{x^2+1}}{3x}$$

$$= \lim_{x\to\infty} \frac{1}{3}\left[\sqrt{1+\frac{5}{x^2}} - \sqrt{1+\frac{1}{x^2}}\right]$$

$$= \frac{1}{3}\left[\sqrt{1+0} - \sqrt{1+0}\right] = 0$$

(*ii*) $$\lim_{x\to\infty} \sqrt{x+1}\left(\sqrt{x+5} - \sqrt{x+4}\right)$$

$$= \lim_{x\to\infty} \frac{\sqrt{x+1}\left(\sqrt{x+5} - \sqrt{x+4}\right)\left(\sqrt{x+5} + \left(\sqrt{x+4}\right)\right.}{\left(\sqrt{x+5} + \sqrt{x+4}\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x+1}+\left(x+5-x-4\right)}{\sqrt{x+5}+\sqrt{x+4}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x+1}}{\sqrt{x+5}+\sqrt{x+4}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x}}}{\sqrt{1+\frac{5}{x}}+\sqrt{1+\frac{4}{x}}}$$ dividing the numerator and the denominator by $x^{1/2}$.

$$= \frac{\sqrt{1+0}}{\sqrt{1+0}+\sqrt{1+0}} = \frac{1}{2}.$$

Type B

This type consists of limits of trigonometric functions. These are evaluated with the help of standard results, simplification by the use of trigonometry formulae and the basic result $\lim_{x\to 0} \frac{\sin x}{x} = 1$ which is to be assumed as the proof is not in the course.

Standard Results. *(i)* $\lim_{x\to\alpha} \sin x = \sin \alpha$, *(ii)* $\lim_{x\to\alpha} \cos \alpha = \cos \alpha$.

Proof. $\lim_{x\to\alpha} \sin x = \sin \alpha$

We have,

$$\lim_{x\to\alpha} (\sin x - \sin\alpha) = \lim_{x\to\alpha} 2 \sin \frac{(x-\alpha)}{2} \cos \frac{x+\alpha}{2}$$

$$= 2 \lim_{x\to\alpha} \sin \frac{x-\alpha}{2} \lim_{x\to\alpha} \cos \frac{x+\alpha}{2}$$

Now, as $x \to \alpha, x-\alpha \to 0 \therefore \frac{x-\alpha}{2} \to 0$

$$\therefore \lim_{x\to\alpha} \sin\left(\frac{x-\alpha}{2}\right) = \lim_{\frac{x-\alpha}{2}\to 0} \sin\left(\frac{x-\alpha}{2}\right)$$

$$= 0 \quad \because \lim_{x\to\alpha} \sin x = 0$$

$$\therefore \lim_{x\to\alpha} \left(\sin x - \sin \alpha\right) = 2 \cdot 0 \cdot \lim_{x\to\alpha} \cos\left(\frac{x+\alpha}{2}\right)$$

$$= 0, \text{ whatever the other limit.}$$

$$\therefore \lim_{x\to\alpha} \sin x - \lim_{x\to\alpha} \sin \alpha = 0$$

$\therefore \lim_{x\to\alpha} \sin x = \sin \alpha$ $\quad\because \lim_{x\to\alpha} \sin \alpha = \sin \alpha$

(*ii*) $\lim_{x\to\alpha} \cos x = \cos \alpha.$

We have,

$$\lim_{x\to\alpha} \left((\cos x - \cos \alpha\right) = \lim_{x\to\alpha} \left[-2 \sin \frac{x-\alpha}{2} \sin \frac{x+\alpha}{2}\right]$$

$$= -2 \lim_{x\to\alpha} \sin \frac{x-\alpha}{2} \lim_{x\to\alpha} \sin \frac{x+\alpha}{2}$$

$$= -2.0. \lim_{x\to\alpha} \sin \frac{x+\alpha}{2}$$

$$= 0$$

$\therefore \lim_{x\to\alpha} \left(\cos x - \cos \alpha\right) = 0$

$\therefore \lim_{x\to\alpha} \cos x - \lim_{x\to\alpha} \cos \alpha = 0$

$\therefore \lim_{x\to\alpha} \cos x = \cos \alpha$ $\quad\because \lim_{x\to\alpha} \cos \alpha = \cos \alpha$

Note : It now follows that

$$\lim_{x\to\alpha} \sin (a x + b) = \sin (a \alpha + b)$$

and $\lim_{x\to\alpha} \cos (ax + b) = \cos (a\alpha + b)$

Case 1. We use standard results in trigonometry to simplify the trigonometric function before putting $x = \alpha$ and evaluating the limit.

Illustrative Examples

Ex. 1. Evaluate $\lim_{x\to \frac{\pi}{4}} \dfrac{2-\sec^2 x}{1-\tan x}$

Sol. $\lim_{x\to \frac{\pi}{4}} \dfrac{2-\sec^2 x}{1-\tan x}$

$= \lim_{x\to \frac{\pi}{4}} \dfrac{2-(1+\tan^2 x)}{1-\tan x}$

$= \quad,, \quad \dfrac{1-\tan^2 x}{1-\tan x}$

$= \quad,, \quad \dfrac{(1-\tan x)(1+\tan x)}{1-\tan x}$ $\qquad x \to \frac{\pi}{4}$

$= \quad,, \quad 1 + \tan x$ $\qquad \therefore \; x \neq \frac{\pi}{4}$

$= \quad,, \quad 1 + 1 \quad = 2.$ $\qquad \text{Tan } x \neq 1$

$\qquad\qquad 1 - \text{Tan} x \neq 0$

Case 2. Use of the result $\lim_{x\to 0} \frac{\sin x^c}{x} = 1$.

Ex. 2. Find $\lim_{x\to 0} \frac{\sin x^\circ}{x}$

Sol. Let $x^\circ = \theta^c$

$\therefore \quad \frac{\pi}{180} x = \theta$. and as $x \to 0$, $\theta \to 0$.

$\therefore \quad \lim_{x\to 0} \frac{\sin x^\circ}{x} = \lim_{\theta\to 0} \frac{\sin \theta}{\theta} \cdot \frac{\pi}{180}$

$= 1 . \frac{\pi}{180} = \frac{\pi}{180}$

Ex. 3. Evaluate the following limits.

(*i*) $\lim_{x\to 0} \frac{1-\cos 2x}{1-\cos 4x}$ (*ii*) $\lim_{x\to \infty} x \sin \frac{\pi}{x}$ (*iii*) $\lim_{x\to y} \frac{\sin 2x - \sin 2y}{x-y}$

Sol. (*i*) $\lim_{n\to 0} \frac{1-\cos 2x}{1-\cos 4x} = \lim_{n\to 0} \frac{2 \sin^2 x}{2 \sin^2 2x}$.

$= \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{(2x)^2}{\sin^2 2x} \times \frac{1}{4}$

$= \frac{1}{4} \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x\to 0} \left(\frac{2x}{\sin 2x}\right)^2$

$= \frac{1}{4} . 1 \cdot 1 = \frac{1}{4}$

$\therefore \quad \lim_{x\to 0} \frac{1-\cos 2x}{1-\cos 4x} = \frac{1}{4}$.

(*ii*) $\lim_{x\to \infty} x \sin \frac{\pi}{x} = \lim_{x\to \infty} \frac{\sin \frac{\pi}{x}}{\frac{\pi}{x}} . \pi$

Put $\theta = \frac{\pi}{x}$. Then *as*, $x \to \infty$, $\theta \to 0$

$\therefore \quad \lim_{x\to \infty} x \sin \frac{\pi}{x} = \pi \lim_{\theta\to 0} \frac{\sin \theta}{\theta}$

$= \pi \cdot 1 = \pi$

(*iii*) $\lim_{x\to y} \frac{\sin 2x - \sin 2y}{x-y}$

$$= \lim_{x\to y} \frac{2\sin(x-y)\cos(x+y)}{x-y}$$

$$= 2.\lim_{x\to y}\frac{\sin(x-y)}{x-y}.\lim_{x\to y}\cos(x+y)$$

Now as $x\to y$, $x-y\to 0$

$$\therefore \lim_{x\to y}\frac{\sin(x-y)}{x-y}=\lim_{x-y\to 0}\frac{\sin(x-y)}{x-y}=1$$

Also. $\lim_{x\to y}\cos(x+y)=\cos(y+y)=\cos 2y.$

$$\therefore \lim_{x\to y}\frac{\sin 2x-\sin 2y}{x-y}=2\cdot 1\cdot\cos 2y.=2\cos 2y$$

Case 3. The characteristics of the set-of examples in this group is that they involve trigonometric functions which are not readily capable of further simplification and/or if $x\to a$ then the factor $x-a$ is present in the function.

Method for evaluating such limits is to put $x-a=\theta$ and evaluate the limit of the new function in θ as $\theta\to 0$ by the methods of earlier cases.

Illustrative Examples

Ex. 4. Evaluate

(*i*) $\lim_{\theta\to\frac{\pi}{3}}\dfrac{\tan\theta-\sqrt{3}}{\frac{\pi}{3}-\theta}$ (*ii*) $\lim_{\theta\to\frac{\pi}{3}}\dfrac{\tan\theta-\sqrt{3}}{\frac{\pi}{3}-\theta}$

(*iii*) $\lim_{x\to 1}\dfrac{\sin(x-1)}{x^2+7x-8}$

Sol. (*i*) $\lim_{n\to\frac{\pi}{6}}\dfrac{1-2\sin x}{\frac{\pi}{6}-x}$

Put $\frac{\pi}{6}-x=\theta$ $\therefore$ $x=\frac{\pi}{6}-\theta$

$\therefore$ as $x\to\frac{\pi}{6}$, $\theta\to 0$

$$\therefore \lim_{x\to\frac{\pi}{6}}\frac{1-2\sin x}{\frac{\pi}{6}-x}$$

$$=\lim_{\theta\to 0}\frac{1-2\sin\left(\frac{\pi}{6}-\theta\right)}{\theta}$$

$$= \lim_{\theta \to 0} \frac{1 - 2\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right)}{\theta}$$

$$= \text{,,} \quad \frac{(1-\cos\theta) + \sqrt{3}\sin\theta}{\theta}$$

$$= \text{,,} \quad \frac{2\sin^2\frac{\theta}{2} + \sqrt{3}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{\theta}$$

$$= \text{,,} \quad \frac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \sqrt{3}\cos\frac{\theta}{2}\right)}{\theta}$$

$$= \text{,,} \quad \frac{\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \sqrt{3}\cos\frac{\theta}{2}\right)}{\frac{\theta}{2}}$$

$$= \lim_{\theta \to 0} \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} \cdot \lim_{\theta \to 0}\left(\sin\frac{\theta}{2} + \sqrt{3}\cos\frac{\theta}{2}\right)$$

$$= \quad 1.\ (0 + \sqrt{3}) = \sqrt{3}$$

(*ii*) $\lim_{\theta \to \frac{\pi}{3}} \frac{\tan\theta - \sqrt{3}}{\frac{\pi}{3} - \theta}$

Put $\frac{\pi}{3} - \theta = h \quad \therefore \quad \theta = \frac{\pi}{3} - h$

As $\theta \to \frac{\pi}{3}$, $h \to 0$

$$\therefore \lim_{\theta \to \frac{\pi}{3}} \frac{\tan\theta - \sqrt{3}}{\frac{\pi}{3} - \theta}$$

$$= \lim_{h \to 0} \frac{\tan\left(\frac{\pi}{3} - h\right) - \sqrt{3}}{h}$$

$$= \text{,,} \quad \frac{\frac{\tan\frac{\pi}{3} - \tan h}{1 + \tan\frac{\pi}{3}\tan h} - \sqrt{3}}{h}$$

$$= \lim_{h\to 0} \frac{\tan\frac{\pi}{3} - \tan h - \sqrt{3}\left(1+\tan\frac{\pi}{3}\tan h\right)}{h\left(1+\tan\frac{\pi}{3}\tan h\right)}$$

$$= \text{,,} \quad \frac{\sqrt{3}-\tan h-\sqrt{3}-3\tan h}{h(1+\sqrt{3}\tan h)} \quad \because \tan\frac{\pi}{3}=\sqrt{3}$$

$$= \text{,,} \quad \frac{-4\tan h}{h(1+\sqrt{3}\tan h)}$$

$$= \text{,,} \quad \frac{-4\tan h}{h} \times \frac{1}{1+\sqrt{3}\tan h} = -4 \ (1)\left(\frac{1}{1+0}\right) = -4$$

$$\therefore \lim_{n\to\frac{\pi}{6}} \frac{1-2\sin x}{\frac{\pi}{6}-x} = -4$$

(*iii*) $\lim_{n\to 1} \dfrac{\sin(x-1)}{x^2+7x-8}$

$$= \lim_{x\to 1} \frac{\sin(x-1)}{(x-1)(x+8)}$$

$$\therefore \lim_{x\to 1} \frac{\sin(x-1)}{(x-1)} \cdot \lim_{x\to 1} \frac{1}{x+8}$$

$$= 1 \cdot \frac{1}{9} = \frac{1}{9}$$

$$\therefore \lim_{x\to 1} \frac{\sin(x-1)}{x^2+7x-8} = \frac{1}{9}$$

Case 4

Ex. 4. Evaluate the following limits;

(*i*) $\lim_{x\to -2} \dfrac{\sin(x+2)}{\sqrt{2x+5}-\sqrt{x+3}}$,

(*ii*) $\lim_{x\to 0} \dfrac{2x^2+3x}{2x+\sin x}$

Sol. (*i*) $\lim_{x\to -2} \dfrac{\sin(x+2)}{\sqrt{2x+5}-\sqrt{x+3}}$

$$= \lim_{x\to -2} \frac{\sin(x+2)}{\sqrt{2x+5}-\sqrt{x+3}} \cdot \frac{\sqrt{2x+5}+\sqrt{x+3}}{\sqrt{2x+5}+\sqrt{x+3}}$$

$$= \lim_{x\to -2} \frac{\sin(x+2)\left[\sqrt{2x+5}+\sqrt{x+3}\right]}{2x+5-x-3}$$

$$= \lim_{x\to -2} \frac{\sin(x+2)\,[\sqrt{2x+5}+\sqrt{x+3}]}{x+2}$$

$$= \lim_{x\to -2} \frac{\sin(x+2)}{(x+2)} \lim_{x\to -2} [\sqrt{2x+5}+\sqrt{x+3}]$$

$$= 1.\ [\sqrt{-4+5}+\sqrt{-2+3}]$$

$$= 2.$$

$$\therefore \lim_{x\to -2} \frac{\sin(x+2)}{\sqrt{2x+5}-\sqrt{x+3}} = 2$$

(*ii*) $\lim_{x\to 0} \dfrac{2x^2+3x}{2x+\sin x}$

$= \lim_{x\to 0} \dfrac{2x+3}{2+\frac{\sin x}{x}}$ dividing the numerator and the denominator by x.

$$= \frac{0+3}{2+1} = 1$$

$$\therefore \lim_{x\to 0} \frac{2x^2+3x}{2x+\sin x} = 1$$

Case 5 :

Ex. 5. *Evaluate.*

(*i*) $\lim_{x\to 0} \dfrac{\sin^{-1} x}{x}$; (*ii*) $\lim_{x\to 0} \dfrac{1}{x} \sin^{-1}[2x\sqrt{1-x^2}]$.

Sol. (*i*) $\lim_{x\to 0} \dfrac{\sin^{-1} x}{x}$

write $\theta = \sin^{-1} x$ $\therefore$ $\sin\theta = x$ and as $x\to 0$, $\theta\to 0$.

$$\therefore \lim_{x\to 0} \frac{\sin^{-1} x}{x} = \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$$

$$\therefore \lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1.$$

(*ii*) $\lim_{x\to 0} \dfrac{1}{x} \sin^{-1}[2x\sqrt{1-x^2}]$

Write $x = \sin\theta$ $\therefore$ $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2x\sqrt{1-x^2}$$

$$\therefore 2\theta = \sin^{-1}[2x\sqrt{1-x^2}].$$

Also as $x\to 0$, $\theta=0$

$$\therefore \lim_{x\to 0} \frac{1}{x} \sin^{-1}[2x\sqrt{1-x^2}] = \lim_{\theta\to 0} \frac{2\theta}{\sin\theta} = 2$$

[It is assumed that $-\frac{\pi}{2} \leqslant 2\theta \leqslant \frac{\pi}{2}$. This is obvious since $\theta\to 0$].

Type C.

This group includes functions of the type $\frac{a^x-1}{x}$ and $\frac{e^x-1}{x}$ or functions which can be brought to this form. We use standard result (6) of § 2·6 to evaluate limits of such functions.

Ex. 1. Evaluate the following limits.

$$(i) \lim_{x\to 0} \frac{p^x-q^x}{x}; \quad (ii) \lim_{x\to 0} \frac{7^x-5^x}{5^x-3^x}; \quad (iii) \lim_{x\to 0} \frac{a^x-1}{\sin x}$$

Sol. (*i*) $$\lim_{x\to 0} \frac{p^x-q^x}{x} = \lim_{x\to 0} \frac{p^x-1+1-q^x}{x}$$

$$= \lim_{x\to 0} \left[\frac{p^x-1}{x} - \frac{q^x-1}{x}\right]$$

$$= \lim_{x\to 0} \left[\frac{p^x-1}{x}\right] - \lim_{x\to 0} \left[\frac{q^x-1}{x}\right].$$

$$= \log_e p - \log_e q$$

$$= \log_e \frac{p}{q}$$

$$\therefore \lim_{x\to 0} \frac{p^x-q^x}{x} = \log_e \frac{p}{q}.$$

(*ii*) $$\lim_{x\to 0} \frac{7^x-5^x}{5^x-3^x} = \lim_{x\to 0} \frac{7^x-5^x}{x} \cdot \frac{x}{5^x-3^x}$$

$$= \lim_{x\to 0} \frac{7^x-5^x}{x} \cdot \frac{1}{\lim\limits_{x\to 0} \frac{5^x-3^x}{x}}$$

$$= \frac{\log_e \frac{7}{5}}{\log_e \frac{5}{3}}$$

(*iii*) $$\lim_{x\to 0} \frac{a^x-1}{\sin x} = \lim_{x\to 0} \frac{a^x-1}{x \cdot \frac{\sin x}{x}}.$$

$$= \frac{\lim\limits_{x\to 0} \frac{a^x - 1}{x}}{\lim\limits_{x\to 0} \frac{\sin x}{x}}$$

$$= \frac{\log_e a}{1} = \log_e a.$$

Type D.

This group involves functions which can be brought to the form $\left(1+\frac{1}{x}\right)^x$ or $(1+u)^{1/u}$ and their limits are obtained by using the standard results (9) and (10) of § 2·6 viz $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ and $\lim\limits_{u\to 0}(1+u)^{1/u} = e$..

Ex. 11. Prove that, : (*i*) $\lim\limits_{x\to-\infty}\left(1-\frac{1}{x}\right)^{-x} = e$

(*ii*) $\lim\limits_{x\to 0}\left(1+\frac{y}{x}\right)^x = e^y$:

(*iii*) $\lim\limits_{x\to 0}(1+\alpha x)^{\frac{1}{x}} = e^{\alpha}$.

Sol. (*i*) $\lim\limits_{x\to-\infty}\left(1-\frac{1}{x}\right)^{-x}$

Write $-x = u$

$\therefore$ if $x\to-\infty$, $u\to+\infty$.

$\therefore \lim\limits_{x\to-\infty}\left(1-\frac{1}{x}\right)^{-x} = \lim\limits_{u\to\infty} = \left(1+\frac{1}{u}\right)^u = e$

(*ii*) $\lim\limits_{x\to\infty}\left(1+\frac{y}{x}\right)^x = \lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{y}}\right)^{\frac{x}{y}\cdot y}$

Write $\frac{x}{y} = u$.

Since y is a a constant, as $x\to\infty$, $x\to\infty$.

$\therefore \lim\limits_{x\to\infty}\left(1+\frac{y}{x}\right)^x = \lim\limits_{u\to\infty}\left(1+\frac{1}{u}\right)^{uy}$.

$$= \lim_{u\to\infty}\left\{\left(1+\frac{1}{u}\right)^u\right\}^y$$

$$= \left\{ \lim_{u \to \infty} \left(1 + \frac{1}{u} \right)^u \right\}$$

$$= (e)^v = e^v$$

$$\therefore \lim_{x \to \infty} \left(1 + \frac{y}{x} \right)^x = e^v$$

(*iii*) $\lim_{x \to 0} (1+\alpha x)^{\frac{1}{x}}$.

Write $x\alpha = u$ $\quad \therefore \quad x = \frac{u}{\alpha}$ and as $x \to 0$, $u \to 0$.

$$\therefore \lim_{x \to 0} (1+\alpha x)^{\frac{1}{x}} = \lim_{u \to 0} (1+u)^{\frac{\alpha}{u}}$$

$$= \lim_{u \to 0} \{(1+u)^{\frac{1}{u}}\}^{\alpha}$$

$$= \{\lim_{u \to 0} (1+u)^{\frac{1}{u}}\}^{\alpha}$$

$$= (e)^{\alpha}.$$

$$\therefore \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e^{\alpha}$$

Ex. 12. Evaluate the following limits.

(*i*) $\lim_{x \to \infty} \left(\frac{x+3}{x-2} \right)^x$; (*ii*) $\lim_{x \to \infty} \left(\frac{2x+3}{2x+1} \right)^{x+1}$

(*iii*) $\lim_{x \to 0} \left(\frac{1+2x}{1-2x} \right)^{\frac{1}{x}}$ (*iv*) $\lim_{x \to \infty} x \log \frac{x+2}{x-1}$.

Sol. (*i*) $\lim_{x \to \infty} \left(\frac{x+3}{x-2} \right)^x = \lim_{x \to \infty} \left(\frac{1+\frac{3}{x}}{1-\frac{2}{x}} \right)^x$

$$= \lim_{x \to \infty} \frac{\left(1+\frac{3}{x}\right)^x}{\left(1-\frac{2}{x}\right)^x}$$

$$= \frac{\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x}{\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x}$$

$$= \frac{e^3}{e^{-2}} = e^5$$

$$\therefore \lim_{x\to\infty}\left(\frac{x+3}{x-2}\right)^x = e^5.$$

(*ii*) $$\lim_{x\to\infty}\left(\frac{2x+3}{2x+1}\right)^{x+1} = \lim_{x\to\infty}\left(\frac{2x+3}{2x+1}\right)^x \cdot \lim_{x\to\infty}\frac{2x+3}{2x+1}$$

$$= \lim_{x\to\infty}\left(\frac{1+\frac{3}{2x}}{1+\frac{1}{2x}}\right)^x \cdot \lim_{x\to\infty}\frac{2+\frac{3}{x}}{2+\frac{1}{x}}$$

$$= \lim_{\to\infty}\frac{\left(1+\frac{3/2}{x}\right)^x}{\left(1+\frac{1/2}{x}\right)^x} \cdot 1$$

$$= \frac{\lim_{x\to\infty}\left(1+\frac{3/2}{x}\right)^x}{\lim_{x\to\infty}\left(1+\frac{1/2}{x}\right)^x}$$

$$= \frac{e^{3/2}}{e^{1/2}} = e$$

$$\therefore \lim_{x\to\infty}\left(\frac{2x+3}{2x+1}\right)^{x+1} = e.$$

(*iii*) $$\lim_{x\to 0}\left(\frac{1+2x}{1-2x}\right)^{\frac{1}{x}} = \frac{\lim_{x\to 0}(1+2x)^{\frac{1}{x}}}{\lim_{x\to 0}(1-2x)^{\frac{1}{x}}}$$

$$= \frac{e^2}{e^{-2}} = e^4.$$

$$\therefore \lim_{x\to 0}\left(\frac{1+2x}{1-2x}\right)^{\frac{1}{x}} = e^4.$$

(*iv*) $$\lim_{x\to\infty} x \log\frac{x+2}{x-1} = \lim_{x\to\infty}\log\left(\frac{x+2}{x-1}\right)^x$$

$$= \lim_{x\to\infty} \log \left(\frac{1+\frac{2}{x}}{1-\frac{1}{x}} \right)^x$$

$$= \lim_{x\to\infty} \log \frac{\left(1+\frac{2}{x}\right)^x}{\left(1-\frac{1}{x}\right)^x}$$

$$= \log \left[\lim_{x\to\infty} \frac{\left(1+\frac{2}{x}\right)^x}{\left(1-\frac{1}{x}\right)^x} \right]$$

$$= \log \left[\frac{\lim\limits_{x\to\infty} \left(1+\frac{2}{x}\right)^x}{\lim\limits_{x\to\infty} \left(1-\frac{1}{x}\right)^x} \right]$$

$$= \log \left[\frac{e^2}{e^{-1}} \right]$$

$$= \log (e^3)$$

$$= 3$$

Type E

In this group we consider functions which are defined by different relations for different intervals into which the f domain is divided. The point at which the definition changes is called a *critical point*. Thus consider the function given as

$$f(x) = x-2 \text{ for } 0 \leqslant x \leqslant 2$$

$$= x \quad \text{for } 2 < x \leqslant 4$$

The point $x = 2$ is a critical point.

At $x = 2, f(2) = 2-2 = 0$, value of the function exists at $x = 2$. To find $\lim\limits_{x\to 2} f(x)$ we will have to consider the limit of $f(x)$ as x tends to 2 through values greater than 2 and separately when $x \to 2$ through values less than 2. If both these limits are the same, then and only then we say that $\lim\limits_{x\to 2} f(x)$ exists and is equal to that common limit. If either limit does not exist finitely or the two exist but are not equal, we say that $\lim\limits_{x\to a} f(x)$ does not exist.

Thus in the function defined above,

when $x < 2$, $\lim_{x \to 2} f(x) = \lim_{x \to 2} (x-2) = 0$

and when $x > 2$, $\lim_{x \to 2} f(x) = \lim_{x \to 2} x = 2$

These are both existing finitely but are not equal.

We say that $\lim_{x \to 2} f(x)$ does not exist.

Now consider the function defined as

$$f(x) = x^2 \text{ for } -1 \leqslant x < 2$$
$$= x+2 \text{ for } 2 \leqslant x < 4$$

when $x < 2$, $\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 4$

when $x > 2$, $\lim_{x \to 2} f(x) = \lim_{x \to 2} x+2 = 4$.

The two limits are the same. Hence in this case we say that $\lim_{x \to 2} f(x)$ exists and is equal to 4.

EXERCISE

(A)

1. Evaluate the following limits :

(*i*) $\lim_{x \to 1} \frac{x^2-6x+5}{x^2+5x-6}$ (*ii*) $\lim_{x \to 2} \frac{x^3-27}{x^2-2x-3}$

(*iii*) $\lim_{x \to 2} \left[\frac{1}{x+2} + \frac{1}{x^2+3x+2}\right]$ (*iv*) $\lim_{x \to 5} \frac{x^{5/2}-5^{5/2}}{x^{3/2}-5^{3/2}}$

(*v*) $\lim_{x \to -1} \left[\frac{1}{x+1} + \frac{2}{x^2-1}\right]$

2. Evaluate :

(*i*) $\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2-x}}{x}$

(*ii*) $\lim_{x \to 3} \frac{x^2-9}{\sqrt{2x+1} - \sqrt{x+4}}$

(*iii*) $\lim_{x \to -1} \frac{\sqrt{3x+7} - \sqrt{x+5}}{x^3+1}$

(*iv*) $\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x\sqrt{4+2x}}$

(*v*) $\lim_{x \to -2} \frac{\sqrt{3x+1} - \sqrt{2x-1}}{\sqrt{x^2+1} - \sqrt{5}}$

(*vi*) $\lim_{x \to 2} \frac{\sqrt{x^2+3x+1} - \sqrt{3x+5}}{\sqrt{2x^2+3} - \sqrt{x^2+11}}$

(*vii*) $\lim_{x\to 2} \dfrac{\sqrt{x-2} - \sqrt{x^3-8}}{\sqrt{x^2-4}}$

(*viii*) $\lim_{x\to 0} \dfrac{\sqrt{3x+2}-\sqrt{x+2}}{\sqrt{2x+2}-\sqrt{x+3}}$

(*ix*) $\lim_{x\to -3} \dfrac{\sqrt{x+5} - \sqrt{3x+11}}{\sqrt{x^2-3} - \sqrt{6}}$

(*x*) $\lim_{x\to 4} \dfrac{\sqrt{x^2+1} - \sqrt{17}}{\sqrt{2x^3+3} - \sqrt{x^3+67}}$.

3. Find the following limits

(*i*) $\lim_{x\to\infty} \dfrac{2x+3}{4x+1}$;

(*ii*) $\lim_{x\to\infty} \dfrac{1-2x}{1+2x}$

(*iii*) $\lim_{x\to\infty} \dfrac{2-x}{2+3x}$;

(*iv*) $\lim_{x\to\infty} \dfrac{x^2+3x+1}{2x^2+x+3}$

(*v*) $\lim_{x\to\infty} \dfrac{3x^2+5x+3}{x^2-2x+7}$;

(*vi*) $\lim_{x\to\infty} \dfrac{5x-x^2}{2+3x^2}$;

(*vii*) $\lim_{x\to\infty} \dfrac{x+3}{3x^2+2x+1}$;

(*viii*) $\lim_{x\to\infty} \dfrac{3x-2}{4+2x-x^2}$;

(*ix*) $\lim_{x\to\infty} \dfrac{3x^2+4x+2}{x+1}$;

(*x*) $\lim_{x\to\infty} \dfrac{x^3+1}{2-3x^3}$.

(*xi*) $\lim_{x\to\infty} \dfrac{x^2+x+1}{3x^3+x^2+1}$

(*xii*) $\lim_{x\to\infty} \dfrac{2x^3+3x^2+2}{x^2-x+3}$

(*xiii*) $\lim_{x\to\infty} \dfrac{3x^3+5x+1}{4-x-3x^3}$

(*xiv*) $\lim_{x\to\infty} \dfrac{x^3+1}{2x-3x^3}$

(*xv*) $\lim_{x\to\infty} \dfrac{x^4+1}{1+2x^2+3x^4}$;

4. Evaluate the following limits :

(*i*) $\lim_{x\to\infty} \dfrac{\sqrt{1+x^2}}{2x+1}$;

(*ii*) $\lim_{x\to\infty} \dfrac{\sqrt{1+x^2}-\sqrt{1+2x^2}}{\sqrt{x^2+3x}+\sqrt{1+x+x^2}}$;

(*iii*) $\lim_{x\to\infty} \dfrac{\sqrt{x^2+3}-\sqrt{x^2-1}}{4x}$;

(*iv*) $\lim_{x\to\infty} \dfrac{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}+\sqrt{x^2+d^2}}$;

(*v*) $\lim_{x\to\infty} \dfrac{\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}+\sqrt{x^2+d^2}}$;

(*vi*) $\lim_{x\to\infty} \sqrt{x+2}\,(\sqrt{x+3}-\sqrt{x+1})$;

(*vii*) $\lim_{x\to\infty} \sqrt{x}\,(\sqrt{2x+1}-\sqrt{2x-3})$;

(*viii*) $\lim_{x\to\infty} \sqrt{x}(\sqrt{x+a}-\sqrt{x})$;

(*ix*) $\lim_{x\to\infty} x(\sqrt{x^2+p}-\sqrt{x^2+q})$;

(*x*) $\lim_{x\to\infty} x\sqrt{x}\,(\sqrt{x^3+2}-\sqrt{x^3+1})$

(B)

5. Evaluate.

(*i*) $\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$; (*ii*) $\lim_{\theta\to 0} \frac{(\operatorname{cosec}\theta - \cot\theta)}{\theta}$

(*iii*) $\lim_{x\to 0} \frac{\sin \lambda x}{x}$; (*iv*) $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx}$;

(*v*) $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$; (*vi*) $\lim_{x\to 0} \frac{\operatorname{cosec} x - \cot x}{x}$;

(*vii*) $\lim_{x\to 0} x \sin\frac{1}{x}$; (*viii*) $\lim_{x\to 0} \frac{1-\cos ax}{x^2}$;

(*ix*) $\lim_{x\to 0} \frac{\cos px - \cos qx}{x^2}$;

(*x*) $\lim_{x\to\infty} x \tan\frac{\lambda}{x}$; (*xi*) $\lim_{x\to\infty} x \sin\frac{\pi}{2x}$ (*xii*) $\lim_{x\to\infty} \frac{\sin^{-1}x}{x}$;

(*xiii*) $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x^3}$;

(*xiv*) $\lim_{\lambda\to 0} \frac{x \sin x}{1-\cos x}$ (*xv*) $\lim_{x\to 0} \frac{3\sin - \sin 3x}{x^3}$.

6. Discuss the behaviour of $\sin\frac{1}{x}$ as $x\to 0$.

7. Evaluate.

(*i*) $\lim_{x\to\alpha} \frac{\sin(x-\alpha)}{(x-\alpha)}$; (*ii*) $\lim_{\theta-\alpha} \frac{\sin\theta - \sin\alpha}{\theta-\alpha}$;

(*iii*) $\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\pi-2x}$; (*iv*) $\lim_{x\to\frac{\pi}{2}} (\sec x - \tan x)$;

(*v*) $\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)}$;

(*vi*) $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\tan\theta-1}{\frac{\pi}{4}-\theta}$; (*vii*) $\lim\limits_{x\to y} \dfrac{\cos x-\cos y}{x-y}$;

(*viii*) $\lim\limits_{x\to 1} \dfrac{1+\cos \pi x}{(1-x)^2}$; (*ix*) $\lim\limits_{\theta\to\frac{\pi}{4}} \dfrac{1-\tan\theta}{1-\sqrt{2}\sin\theta}$;

(*x*) $\lim\limits_{x\to\frac{\pi}{6}} \dfrac{1-\sqrt{3}\tan x}{\pi-6x}$;

(*xi*) $\lim\limits_{x\to 2} \dfrac{\sin(x-2)}{x^2+4x-12}$;

8. Evaluate

(*i*) $\lim\limits_{x\to 0} \dfrac{\sin x^\circ}{x}$; (*ii*) $\lim\limits_{\theta\to 0} \dfrac{\sin mx^\circ}{\sin nx^\circ}$;

(*iii*) $\lim\limits_{\theta\to 0} \dfrac{\tan\theta^\circ}{\theta}$; (*iv*) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{x^2(1+\cos x)}$;

(*v*) $\lim\limits_{x\to\theta} \dfrac{\sin x-\sin\theta}{\sqrt{x}-\sqrt{\theta}}$;

(*vi*) $\lim\limits_{x\to 0} \dfrac{x^2+2x}{x+\sin 2x}$; (*vii*) $\lim\limits_{x\to\infty} \dfrac{x^2+2x\sin x}{\sin x+3x^2}$;

(*viii*) $\lim\limits_{x\to 1} \dfrac{x^2-3x+2}{x^2-x+\sin(x-1)}$;

(*ix*) $\lim\limits_{x\to -1} \dfrac{\sin(x+1)}{\sqrt{2x+3}-\sqrt{x+2}}$;

(*x*) $\lim\limits_{x\to a} \dfrac{\sin x-\sin a}{\sqrt{4x+a}-\sqrt{x+4a}}$,

9. Evaluate.

(*i*) $\lim\limits_{x\to 0} \dfrac{\tan^{-1}x}{x}$; (*ii*) $\lim\limits_{x\to a} \dfrac{\sin^{-1}x-\sin^{-1}a}{x-a}$;

(*iii*) $\lim\limits_{x\to 0} \dfrac{1}{x}\sin^{-1}(3x-4x^3)$;

(*iv*) $\lim\limits_{x\to 0} \dfrac{1}{x}\tan^{-1}\dfrac{2x}{1-x^2}$;

(C)

10. Evaluate.

(*i*) $\lim\limits_{x\to 0} \dfrac{3^x-1}{x}$; (*ii*) $\lim\limits_{x\to 0} \dfrac{7^x-5^x}{x}$;

(*iii*) $\lim_{x\to 0} \frac{a^{\sin x}-1}{\sin x}$; (*iv*) $\lim_{x\to 0} \frac{5^x-3}{3^x-1}$;

(*v*) $\lim_{x\to 0} \frac{a^x-1}{\sin x}$; (*vi*) $\lim_{x\to \frac{\pi}{2}} \frac{e^{\frac{\pi}{2}-x}-1}{\cos x}$;

(*vii*) $\lim_{y\to 1} \frac{y-1}{\log y}$; (*viii*) $\lim_{x\to 0} \frac{e^x-e^{-x}}{e^{3x}-e^{-3x}}$;

(*ix*) $\lim_{x\to 0} \frac{e^x+e^{-x}+2}{x \sin x}$

(D)

11. Evaluate.

(*i*) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{2x}$; (*ii*) $\lim_{x\to\infty} \frac{2^x-3^x}{2^x+3^x}$;

(*iii*) $\lim_{x\to\infty} \frac{e^x+e^x}{e^x-e^{-x}}$; (*iv*) $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$;

(*v*) $\lim_{x\to\infty} \left(1-\frac{2}{x}\right)^x$ (*vi*) $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x$;

(*vii*) $\lim_{x\to\infty} \left(\frac{x-1}{x+1}\right)^x$; (*viii*) $\lim_{x\to\infty} \left(\frac{x-3}{x+2}\right)^{x+1}$;

(*ix*) $\lim_{x\to\infty} \left(\frac{3x+1}{3x-1}\right)^x$; (*x*) $\lim_{x\to\infty} \left(\frac{2x+3}{3x-2}\right)^{x+1}$;

(*xi*) $\lim_{x\to\infty} \left(\frac{x-2}{2x+1}\right)^{x+1}$; (*xii*) $\lim_{u\to 0} (1+2u)^{\frac{1}{u}}$;

(*xiii*) $\lim_{x\to 0} \left(\frac{1+x}{1-x}\right)^{\frac{1}{x}}$; (*xiv*) $\lim_{x\to 0} \left(\frac{2x+1}{1-2x}\right)^{\frac{1}{x}}$;

(*xv*) $\lim_{u\to 0} \left(\frac{2+u}{2-u}\right)^{\frac{1}{u}}$.

12. Evaluate

(*i*) $\lim_{x\to 0} x \log\left(1+\frac{2}{x}\right)$;

(*ii*) $\lim_{x\to 0} (x+1) \log\left(\frac{x+3}{x}\right)$;

(*iii*) $\lim_{x\to\infty} x^2 \log\left(\frac{x^2-1}{x^2}\right)$;

(*iv*) $\lim_{x\to\infty} x \log \frac{x+2}{x-2}$; (*v*) $\lim_{u\to\infty} \frac{1}{u} (\log (1+2u)$;

(*vi*) $\lim_{x\to 0} \frac{1}{x} \log \left(\frac{1+2x}{1-2x} \right)$;

(*vii*) $\lim_{x\to\infty} \frac{x+1}{x} \log (1-3x)$;

(*viii*) $\lim_{x\to 0} \frac{1}{x} \log \left(\frac{2+3x}{2-3x} \right)$.

(**E**)

13. Evaluate.

(*i*) $\lim_{x\to 2} f(x)$ where $f(x)=x^2-x+2$ where $x<2$

$=1+\frac{3}{x-1}$ when $x \geqslant 2$

(*ii*) $\lim_{x\to 1} f(x)$ where $f(x)=x+1$ when $x<1$

$=3x-2$ when $x \geqslant 1$

(*iii*) $\lim_{x\to 0} f(x)$ where $f(x)=\frac{1}{x}$ when $x < 0$

$=x+1$ when $x \geqslant 0$

ANSWERS

(**A**)

1. (*i*) $-\frac{4}{7}$; (*ii*) $\frac{27}{4}$; (*iii*) -1; (*iv*) $\frac{25}{3}$; (*v*) $-\frac{1}{2}$.

2. (*i*) $\frac{1}{\sqrt{2}}$; (*ii*) $\frac{3}{7}\sqrt{7}$; (*iii*) $\frac{1}{6}\sqrt{2}$; (*iv*) $\frac{1}{8}\sqrt{2}$:

(*v*) $-\frac{1}{8}(\sqrt{3}-1)$; (*vi*) $\frac{1}{33}\sqrt{209}$ (*vii*) $\frac{1}{2}(1-2\sqrt{3})$;

(*viii*) $\frac{1}{2}\sqrt{6}$ (*ix*) $\frac{1}{12}\sqrt{2}$; (*x*) $\frac{8}{17}\sqrt{2227}$.

3. (*i*) $\frac{1}{2}$; (*ii*) -1, (*iii*) $-\frac{1}{3}$; (*iv*) $\frac{1}{2}$; (*v*) 3;

(*vi*) $-\frac{1}{3}$; (*vii*) 0; (*viii*) 0; (*ix*) ∞; (*x*) $-\frac{1}{3}$; (*xi*) 0;

(*xii*) ∞; (*xiii*) -1; (*xiv*) $-\frac{1}{3}$; (*xv*) $\frac{1}{3}$.

4. (*i*) $\frac{1}{2}$; (*ii*) $\frac{1}{2}(1-\sqrt{2})$; (*iii*) 0, (*iv*) $\frac{a^2-b^2}{c^2-d^2}$; (*v*) 1;

(*vi*) 1; (*vii*) $\sqrt{2}$; (*viii*) $\frac{a}{2}$ (*ix*) $\frac{p-q}{2}$ (*x*) $\frac{1}{2}$.

(B)

5. (*i*) 1 ; (*ii*) $\frac{1}{2}$; (*iii*) λ ; (*iv*) $\frac{m^2}{n^2}$, (*v*) $-\frac{1}{2}$; (*vi*) $\frac{1}{2}$;
(*vii*) 1 ; (*viii*) $\frac{a^2}{2}$: (*ix*) $\frac{q^2-p^2}{2}$; (*x*) λ ; (*xi*) $\frac{2}{\pi}$;
(*xii*) 1 ; (*xiii*) 1 ; (*xiv*) 2 ; (*xv*) 2 ; (*xvi*) 4.

7. (*i*) 1 ; (*ii*) $\cos\alpha$, (*iii*) $\frac{1}{2}$; (*iv*) 0 ; (*v*) 0 ; (*vi*) -2 ;
(*vii*) $-\sin y$; (*viii*) $\frac{1}{2}$; (*ix*) 2 ; (*x*) $\frac{1}{9}\sqrt{3}$; (*xi*) $\frac{1}{8}$.

8. (*i*) $\frac{\pi}{180}$; (*ii*) $\frac{m}{n}$ (*ii*) $\frac{\pi}{180}$; (*iv*) $\frac{1}{2}$; (*v*) $2\sqrt{\theta}\cos\theta$;
(*vi*) $\frac{2}{3}$; (*vii*) $\frac{1}{3}$; (*viii*) $-\frac{1}{2}$; (*ix*) 2 ;
(*x*) $\frac{2}{3}\sqrt{5a}\cos a$.

9. (*i*) 1 ; (*ii*) $\frac{1}{\sqrt{1-a^2}}$; (*iii*) 3 ; (*iv*) 2.

10. (*i*) $\log 3$; (*ii*) $\log\frac{7}{5}$; (*iii*) $\log a$; (*iv*) $\frac{\log\frac{5}{3}}{\log 3}$;
(*v*) $\log a$ (*vi*) 1 ; (*vii*) 1 ; (*viii*) $\frac{1}{3}$; (*ix*) 1.

11. (*i*) e^2 ; (*ii*) -1 ; (*iii*) 1 ; (*iv*) e^{-1} ; (*v*) e^{-2} ; (*vi*) e^3 ;
(*vii*) e^{-2} ; (*viii*) e^{-5} ; (*ix*)$e^{2/3}$; (*x*) 0 ; (*xi*) 0 ; (*xii*) e^2 ;
(*xiii*) e^2 ; (*xiv*) e^4 ; (*xv*) e.

12. (*i*) 2 ; (*ii*) 3 ; (*iii*) -1, (*iv*) 4 ; (*v*) 2 ; (*vi*) 4 ;
(*vii*) -3 ; (*viii*) 3.

13. (*i*) 4 ; (*ii*) does not exist ; (*ii*) does not exist.

Continuity

2·8. Continuity at a point

We now introduce a very important concept in calculus. It may be said to be a mathematical formulation of the condition under which the graph of a function is a continuous unbroken curve.

Examine the graphs of the following functions.

Ex. 1. $f(x)=x-2\ 0\leqslant x\leqslant 4$

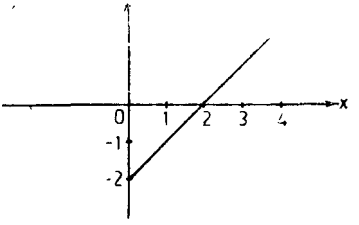

Fig. 2·6

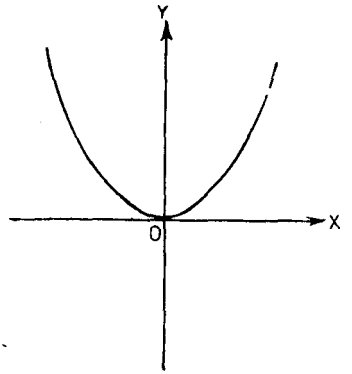

Fig. 2·7.

Ex. 2. $f(x)=x^2 \quad -1 \leqslant x \leqslant 1$

Ex. 3. $f(x)=|x| \quad -3 \leqslant x \leqslant 3$

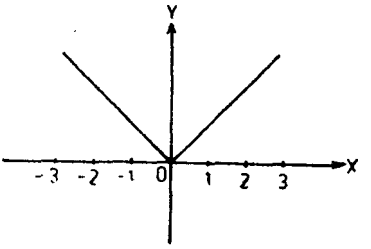

Fig. 2·8

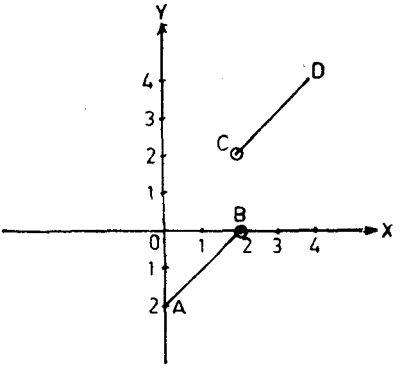

Fig. 2·9.

Ex. 4. $f(x)=x-2 \quad 0 \leqslant x \leqslant 2$
$=x \qquad 2 \leqslant x \leqslant 4$

Ex. 5. $f(x) = \text{Tan } x \quad 0 \leqslant x, \leqslant \pi, \; x \neq \frac{\pi}{2}$

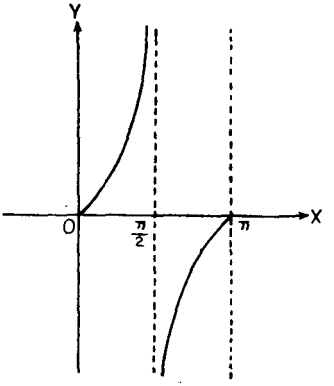

Fig. 2·10.

You will observe that,

(*i*) The graph of the functions in examples 1, 2 and 3 are unbroken curves.

(*ii*) The graphs of the functions in examples 4 and 5 are broken at $x = 2$ and $x = \frac{\pi}{2}$ respectively.

We say that the functions in examples 4 and 5 are not continuous or are discontinuous at $x = 2$ and $x = \frac{\pi}{2}$ respectively. On the other hand the functions in examples 1, 2 and 3 are said to be continuous at every point of their domain.

Now let us examine the behaviour of a function in the neighbourhood of a point at which it is continuous.

Let $f(x) = x^2$. As seen above, this function has a continuous unbroken graph. It is continuous at every point.

Let us consider its behaviour in the neighbourhood of $x = 2$.

We prepare the following table :

x	$f(x)$
1·9	3·61
1·99	3·9601
1·999	3·996001
2	4
2·001	4·004001
2·01	4·0401
2·1	4·41

Observe that,

(*i*) As x approaches 2 from the left or the right the values of the function approach closer and closer to 4. Also, $f(2) = 4$. *i.e.* the values of the function approach $f(2)$.

(*ii*) Also, $\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 4$.

Thus, at $x = 2$ we have,

$$\lim_{x \to 2} f(x) = 4 = f(2).$$

$\therefore$ at $x = 2$ which is a point of continuity of the function in the sense that its graph is unbroken at $x = 2$, we have,

$$\lim_{x\to 2} f(x) = 4 = f(2).$$

Consider again, $f(x) = x-2 \quad 0 \leqslant x \leqslant 4$

We have, $\lim_{x\to 2} f(x) = \lim_{x\to 2} (x-2) = 0$

Also, $f(2) = 0$.

$$\therefore \lim_{x\to 2} f(x) = f(2)$$

We see that at $x = 2$, the graph of the function is unbroken.

On the other hand for $f(x) = \tan x$, we find that the function is not defined at $x = \frac{\pi}{2}$.

However for the function f defined in ex. 4, $f(2)$ exists and is equal to 2.

But when $x \to 2$ through values less than 2, $f(x)$ approaches 0 while when $x \to 2$ through values greater than 2, $f(x)$ approaches 2. Hence $\lim_{x\to 2} f(x)$ does not exist as there is no common limit.

Here again the graph of the function is broken at $x = 2$.

These conditions lead us to the following definition of the continuity of a function at a point.

Def. 1. A function f is said to be continuous at a point c of its domain if $\lim_{x\to c} f(x) = f(c)$.

OR

Def. 2. A function f is said to be continuous at a point c of its domain if given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x)-f(c)| < \epsilon \quad \text{for all} \quad |x-c| < \delta.$$

Note that the two definitions are equivalent and for a function f to be continuous at $x = c$

(*i*) $f(c)$ must exist and be finite *i.e.* $f(c)$ must not be infinite, indeterminate or imaginary.

(*ii*) $\lim_{x\to c} f(x)$ also must exist and

(*iii*) the two must be equal.

If $f(x)$ is not continuous at $x = a$, it is said to be *discontinuous* at $x = a$

Let us now apply the definition to a few functions.

Illustrative Examples.

Ex. 1. $f(x) = 3x+2$. Examine for continuity of f at $x = 1$.

Sol. We have,

$f(1) = 3 \cdot 1 + 2 = 5$

Also, $\lim_{x \to 1} f(x) = \lim_{x \to 1} (3x+2) = 5$

$\therefore \lim_{x \to 1} f(x) = f(1)$

$\therefore$ The function is continuous at $x = 1$.

Observe that the graph of the function is an unbroken line.

Ex. 2. If, $f(x) = \frac{1}{x^2}$, $x \neq 0$, $f(0) = 0$; examine the continuity of f at $x = 0$.

Sol. We have, $f(0) = 0$

Now, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty$

$\therefore \lim_{x \to 0} f(x) \neq f(0)$

$\therefore$ The function is not continuous at $x = 0$.

Ex. 3. If, $f(x) = \frac{x^2-9}{x-3}$ for $x \neq 3$

$= 1$ for $x = 3$

discuss the continuity of f at $x = 3$.

Sol. Here, $f(3) = 1$

Now, $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3}$

$= \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)}$

$= \lim_{x \to 3} (x+3)$ $\because x \to 3$, $x \neq 3$, and $x-3 \neq 0$.

$= 6$.

$\therefore \lim_{x \to 3} f(x) = 6 \neq f(3)$

$\therefore$ The function is discontinuous at $x = 3$.

Note. In this case, the limit and the value both exist but are unequal.

Ex. 4. $f(x) = \frac{\sin x}{x}$, $x \neq 0$ and $f(0) = 1$.

Show that the function is continuous at $x = 0$

Sol. $f(0) = 1$. Also, $\lim_{x \to 0} \frac{\sin x}{x} = 1 = f(0)$.

$\therefore$ the function is continuous at $x = 0$

Ex. 5. Show that $f(x) = \sin x$ is continuous at $x=c$ where c is any real number.

Sol. $f(x) = \sin x$

$\therefore \quad f(c) = \sin c$

Also, $\lim_{x\to c} f(x) = \lim_{x\to c} \sin x$

$= \sin c$

$\therefore \quad \lim_{x\to c} f(x) = f(c).$

$\therefore$ $\sin x$ is continuous at $x = c$.

OR $\quad |f(x)-f(c)| < \epsilon$

if $\quad |\sin x - \sin c| < \epsilon$

i.e. if $\quad \left|2\cos\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)\right| < \epsilon$

i.e. if $\quad \left|2\cdot 1.\left(\frac{x-c}{2}\right)\right| < \epsilon \qquad \because \cos\left(\frac{x-c}{2}\right) \leqslant 1$

and $\sin\left(\frac{x-c}{2}\right) < \left(\frac{x-c}{2}\right)$

i.e. if $\quad |x-c| < \epsilon$

Taking $\delta = \epsilon$ we get by def. 2 that $\sin x$ is continuous at $x = c$.

Note that the graph of $\sin x$ is a continuous curve.

Ex. 6. Discuss the continuity of f at $x = 2$ if

$$f(x) = x^2 - x + 2 \qquad \text{when } x < 2$$

$$= 1 + \frac{3}{x-1} \qquad \text{when } x \geqslant 2.$$

Sol. We have $f(2) = 1 + \frac{3}{2-1} = 1+3 = 4$

$\therefore$ value of f exists at $x = 2$.

Since $x = 2$ is a critical point, we have to find $\lim_{x\to 2} f(x)$ through values of x less than 2 and also through values of x greater than 2.

when $x < 2$, $\lim_{x\to 2} f(x) = \lim_{x\to 2} (x^2 - x + 2) = 4 - 2 + 2 = 4$

When $x > 2$, $\lim_{x\to 2} f(x) = \lim_{x\to 2} 1 + \frac{3}{x-1} = 1 + \frac{3}{2-1} = 4$

$\therefore \lim_{x \to 2} f(x) = 4 = f(2)$

$\therefore$ f is continuous at $x = 2$.

Ex. 7. Discuss the continuity of f at $x = 1$ if

$$f(x) = x-2 \quad \text{for } x < 1$$
$$= 4x-1 \quad \text{for } x \geqslant 1$$

Sol. We have $f(1) = 4(1)-1 = 3$

$\therefore$ value of f exists at $x = 1$.

Now $x = 1$ is a critical point.

When $x < 1$, $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x-2) = 1-2 = -1$

When $x > 1$, $\lim_{x \to 1} f(x) = \lim_{x \to 1} (4x-1) = 4(1)-1 = 3$

These are not equal $\therefore \lim_{x \to 1} f(x)$ does not exist.

$\therefore$ f is discontinuous at $x = 1$.

2·9. Continuity in an interval.

It is now simple to extend the idea of continuity of a function at a point to that of continuity in an interval. We define as follows.

Def. 1. A function f is said to be continuous in an interval $[a, b]$ if it is continuous at every point of that interval.

Note. To discuss continuity of a function in an interval, we discuss its continuity at a general point which is not a critical point and separately at every critical point of the interval.

Illustrative Examples

Ex. 1. Examine for continuity in $[-1, 1]$, of the function f defined as,

$$f(x) = 3x-2 \qquad x \neq 0$$
$$= 1 \qquad x = 0$$

Sol. Let $c \in [1,-1]$ and $c \neq 0$.

Then $f(c) = 3c-2$.

Also $\lim_{x \to c} f(x) = \lim_{x \to c} (3x-2)$

$$= 3c-2$$

$\therefore \lim_{x \to c} f(x) = f(x)$ when $c \neq 0$.

$\therefore$ The function is continuous at $x = c$ if $c \neq 0$

Now, $f(0) = 1$

and $\lim_{x \to 0} f(x) = \lim_{x \to 0} (3x-2)$

$= 3 \cdot 0 - 2 = -2$

$\therefore \lim_{x \to 0} f(x) = -2 \neq f(0)$.

$\therefore$ The function is not continuous at $x = 0$

$\therefore$ f is continuous at every point of $[-1, 1]$ except at $x = 0$.

Ex. 2. Examine the following functions for continuity in the intervals indicated.

(*i*) $f(x) = \dfrac{x^2-3x+2}{x-1}$ $\quad x \neq 1$

$= 0 \quad x = 1$ in $[0, 2]$.

(*ii*) $f(x) = \dfrac{x^2-4}{x-2}$ for $x \neq 2$ in $[1, 3]$.

$= 4$ for $x = 2$

Sol. (*i*) Let $c \in [0, 2]$ and $c \neq 1$

$\therefore f(c) = \dfrac{c^2-3c+2}{c-1}$

$= \dfrac{(c-2)(c-1)}{c-1}$

$= c-2 \quad \because c \neq 1$

Also, $\lim_{x \to c} f(x) = \lim_{x \to c} \dfrac{x^2-3x+2}{x-1}$

$= \lim_{x \to c} \dfrac{(x-1)(x-2)}{(x-1)}$

$= \lim_{x \to c} (x-2) \quad \because x \neq 1$

$= c-2$

$\therefore \lim_{x \to c} f(x) = c-2 = f(c)$

$\therefore$ f is continuous at $x = c$ if $c \neq 1$.

Now, $f(1) = 0$

and $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{x^2-3x+2}{x-1}$

$= \lim_{x \to 1} \dfrac{(x-2)(x-1)}{(x-1)}$

$= \lim_{x \to 1} (x-2) \quad \because x \to 1, x \neq 1$ and $x-1 \neq 0$

$= 1-2 = -1$

$\therefore \lim_{x \to 1} f(x) = -1 \neq f(1)$

$\therefore$ The function is discontinuous at $x = 1$.

$\therefore$ The function is discontinuous at $x = 1$ but it is continuous at all other points of [0, 2].

(*ii*) $f(x) = \dfrac{x^2-4}{x-2}$ for $x \neq 2$

$= 4$ for $x = 2$ in [1, 3]

Let $c \in [1, 3]$ and $c \neq 2$.

$\therefore f(c) = \dfrac{c^2-4}{c-2}$

$= c + 2$ for $c \neq 2$.

$\therefore \lim_{x \to c} f(x) = \lim_{x \to c} \dfrac{x^2-4}{x-2}$

$= \dfrac{c^2-4}{c-2}$ for $c \neq 2$

$= c + 2$

$\therefore \lim_{x \to c} f(x) = f(c)$

$\therefore$ The function is continuous at $x = c$ if $c \neq 2$.

Now, $f(2) = 4$

and $\lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{x^2-4}{x-2}$

$= \lim_{x \to 2} \dfrac{(x-2)(x+2)}{(x-2)}$

$= \lim_{x \to 2} (x+2)$ $\because x \to 2, x \neq 2$ and $x-2 \neq 0$

$= 4.$

$\therefore \lim_{x \to 2} f(x) = 4 = f(2).$

$\therefore$ The function is continuous at $x = 2$ also.

$\therefore$ The function is continuous in [1, 3].

Observe that, in all the examples we have to give special attention to the values of x at which the function is defined differently or where the definition changes. These are the points at which it is likely to change its behaviour suddenly, so that a break may appear in its graphs. It is then a point of discontinuity of the function.

Ex. 3. Examine the following functions for continuity in the intervals indicated.

(*i*) $f(x) = \dfrac{1}{x^2} \qquad x \neq 0$

$= 0$ for $x = 0$ in $[-1, 1]$

(*ii*) $f(x) = \dfrac{x^2+3x+2}{(x-1)^2}$ for $x \neq 1$

$= 1$ for $x = 1$ in $[0, 2]$

Sol. (*i*) $f(x) = \dfrac{1}{x^2}$, $x \neq 0$ and $f(0) = 0$.

Let $c \in [-1, 1]$ and $c \neq 0$.

$\therefore \quad f(c) = \dfrac{1}{c^2}$

Also, $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \dfrac{1}{x^2}$

$= \dfrac{1}{c^2} \qquad \because \quad c \neq 0$

$\therefore \quad \lim\limits_{x \to c} f(x) = f(c)$ if $c \neq 0$

$\therefore$ The function is continuous at $x = c$ if $c \neq 0$.

Now we consider $c = 0$.

We have $f(c) = 0$. Let us observe the behaviour of the function in the neighbourhood of $x = 0$. We prepare the following table

x :	$-0{\cdot}1$	$-0{\cdot}01$	$-0{\cdot}001$	0	$0{\cdot}0001$	$0{\cdot}01$	$0{\cdot}1$
$f(x)$:	10^2	10^4	10^6	0	10^6	10^4	10^2

As x approaches 0 we see that $f(x)$ becomes larger and larger, This may be expressed as $f(x) \to \infty$ as $x \to 0$.

$\therefore \quad \lim\limits_{x \to 0} f(x) = \infty$ and $f(0) = 0$

$\therefore$ The function is discontinuous at $x = 0$. At all other points of $[-1, 1]$ it is a continuous function.

(*ii*) $f(x) = \dfrac{x^3+3x+2}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$ in $[0, 2]$.

As in (*i*) above it can be verified that for $x \neq 1$ the function is continuous in $[0, 2]$.

At $x = 1$, $f(1) = 1$.

We prepare the following table to observe the behaviour of the function in the neighbourhood of $x = 1$.

x :	0·9	0·99	0·999	1	1·001	1·01	1·1
$f(x)$:	551	58501	5995001	1	6005001	63001	701

We see that as x approaches 1, $f(x)$ becomes larger and larger so that in this case also,

$$\lim_{x \to 1} f(x) = \infty \text{ and } f(1) = 1.$$

$\therefore$ The function is discontinuous at $x = 1$, at every other point of $[0, 2]$ it is continuous.

Note that at every point where the denominator of a function tends to 0 but the numerator remains finite and $\neq 0$, the function will be discontinuous.

Ex. 4. Discuss the continuity of $f(x)$ in $[2, 4]$ where

$$f(x) = 4x - 1 \quad \text{for } 2 \leqslant x \leqslant 3$$
$$= x^2 - 5 \quad \text{for } 3 < x \leqslant 4$$

Sol. Let $c \in [2, 4]$ and $c \neq 3$

If $2 \leqslant c < 3$ we have

$$f(c) = 4c - 1 \quad \text{and}$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (4x - 1) = 4c - 1$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Hence $f(x)$ is continuous at $x = c$ and hence in the interval $2 \leqslant x \leqslant 3$ ...(3

If $3 < c \leqslant 4$, we have

$$f(c) = c^2 - 5 \text{ and}$$

$$\lim_{x \to c} f(c) = \lim_{x \to c} (x^2 - 5) = c^2 - 5$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

$\therefore$ $f(x)$ is continuous at $x = c$ and hence in the interval $3 < x \leqslant 4$ ...(2)

Now $x = 3$ is a critical point. We have

$$f(3) = 4 \times 3 - 1 = 11$$

$\therefore$ Value of $f(x)$ exists at $x = 3$

When $x < 3$, $\lim_{x\to 3} f(x) = \lim_{x\to 3} (4x-1)$

$= 4\times 3-1 = 11$

When $x > 3$, $\lim_{x\to 3} f(x) = \lim_{x\to 3} (x^2+5)$

$= 9+5 = 14$

These are not equal.

$\lim_{x\to 3} f(x)$ does not exist

$\therefore$ f is not continuous at $x = 3$ ...(3)

From (1), (2) and (3) we get that f is continuous in [2, 4] except at $x = 3$.

2·10. Algebra of Continuous Functions

As in the case of finding limits of functions, it is not convenient to apply every time, the definition of continuity of a function. We therefore rely on some theorems and the continuity of some basic functions to deduce the continuity or otherwise of a large class of functions.

Let f and g be two functions continuous at $x=c$. Then the following theorems can be proved.

Theorem 1 : The function $f+g$ is continuous at $x=c$.

Theorem 2 : The function $f-g$ is continuous at $x=c$.

Theorem 3 : The function $f.\ g$ is continuous at $x=c$.

Theorem 4 : The function $f\div g$ is continuous at $x=c$ if $g(x)\neq 0$ in the domain

The proofs of these theorems are omitted as they are beyond the scope of the syllabus.

2·11. Basic Continuous Functions

1. $f(x)=x^n$ is continuous at all points if $n \geqslant 0$. It is discontinuous at $x=0$ if $n < 0$.

2. $f(x)=\sin x$ and $g(x)=\cos x$ are continuous everywhere.

3. $f(x)=a^x$ is continuous everywhere.

4. $f(x)=\log x$ is a continuous function for $x > 0$.

5. If f is a continuous function of u and u is a continuous function of x then f is a continuous function of x.

2·12. With the help of the theorems and the basic functions we now deduce a few results by way of illustrations :

1. Every polynomial is a continuous function.

Consider

$$f(x)=x^3+3x^2+2x+5$$

Each term on the r.h.s. is a continuous function of x. Hence by the first theorem f is a continuous function.

In general if,

$$P(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n,$$

n being a positive integer, by a similar argument it follows that it is a continuous function.

2. Let $f(x)=\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x. Then f is continuous everywhere expect at those points where $Q(x)=0$ and $P(x)\neq 0$. The points at which $P(x)$ and $Q(x)$ are both 0, will have to be considered specially.

Thus,

$$f(x)=\frac{x^2+3x+5}{x^2-3x+2},\ x\neq 1,\ 2$$
$$=0 \quad \text{for } x=1 \text{ and } x=2.$$

is discontinuous at $x=1$ and $x=2$, since at these points,

$$x^2-3x+2=0$$

but $$x^2+3x+5\neq 0.$$

On the other hand if

$$g(x)=\frac{x^2+4x-5}{x^2-3x+2},\ x\neq 1 \text{ and } x\neq 2$$
$$=-6 \quad \text{for } x=1$$
$$=0 \quad \text{for } x=2$$

We have, $g(1)=-6$ and

$$\lim_{x\to 1} g(x)=\lim_{x\to 1}\frac{(x-1)(x+5)}{(x-1)(x-2)}$$
$$=\lim_{x\to 1}\frac{(x+5)}{(x-2)} \quad \because\ x\to 1 \text{ and } x\neq 1$$
$$=-6.$$

$$\therefore \quad \lim_{x\to 1} g(x)=g=(1)$$

$\therefore$ g is continuous at $x=1$.

Observe that at $x=1$

$x^2-3x+2=0$ and $x^2+4x-5=0$

But, at $x=2$,

$x^2-3x+2=0$ and $x^2+4x-5\neq 0$.

and g is discontinuous at $x=2$.

3. Using (1) above and (5) in the last section we now obtain that every polynomial in sin x and cos x is a continuous function.

Thus,

$$f(x)=2\sin^2 x+3\sin x+5$$

is continuous every where.

Similarly, $a^x \sin 2x+\cos(2x+3)$ is a continuous function.

The student should himself construct some continuous functions in the manner indicated above.

EXERCISE

1. Examine the following functions for continuity at $x=0$.

(*i*) $f(x)=\dfrac{\sin x}{x}$, $x\neq 0$ $\quad=1$ for $x=0$

(*ii*) $f(x)=x\sin\dfrac{1}{x}$ for $x\neq 0$ $\quad=1$ for $x=0$

(*iii*) $f(x)=x\cos\dfrac{1}{x}$ for $x\neq 0$ $\quad=0$ for $x=0$

(*iv*) $f(x)=\dfrac{\sin x^3}{x^2}$ for $x\neq 0$ $\quad=0$ for $x=0$

(*v*) $f(x)=\dfrac{\tan 3x}{2x}$ for $x\neq 0$ $\quad=3/2$ for $x=0$

2. Prove that,

(*i*) $f(x)=\dfrac{x^2-9}{x^2-4x+3}$ for $x\neq 3$ $\quad=3$ for $x=3$.
is continuous at $x=3$.

(*ii*) $f(x)=\dfrac{x^2+3x+2}{x^2-1}$ for $x\neq -1$ $\quad=-1$ for $x=-1$.
is continuous at $x=-1$.

(*iii*) $f(x)=\dfrac{x^2-4x+3}{2x^2-3x+1}$ when $x\neq 1$ $\quad=1$ when $x=1$.
is continuous at $x=1$.

(*iv*) $f(x)=\dfrac{\sin x}{x^2+2x}$ for $x=0$ $\quad=\frac{1}{2}$ for $x=0$.
is continuous at $x=0$.

(*v*) $f(x)=\dfrac{\sqrt{x+2}-2}{x^2-4}$, $x\neq 2$ $\quad=0$ for $x=2$
is continuous at $x=2$.

(*vi*) $f(x)=\dfrac{x}{\sqrt{2x+1}-\sqrt{x+1}}$, $x\neq 0$,
$f(0)=1$
is discontinuous at $x=0$.

(*vii*) $f(x)=\dfrac{a^x-1}{x}$, $x\neq 0$ and $f(0)=0$ is discontinuous at $x=0$.

(*viii*) $f(x)=\dfrac{3^x-2^x}{x}$ for $x\neq 0$ and $f(0)=\log\dfrac{3}{2}$ is continuous at $x=0$.

3. In the following f is not defined at the points mentioned. Define f at these points so that it becomes continuous at these points.

(*i*) $f(x)=\dfrac{4x^2-9}{2x-3}$ for $x\neq\dfrac{3}{2}$ (*ii*) $f(x)=\dfrac{x^2+5x+4}{x^2+3x+2}$ for $x\neq -1$

(*iii*) $f(x)=x\sin\dfrac{1}{x}$ for $x\neq 0$ (*iv*) $f(x)=\dfrac{a^x-1}{x}$ for $x\neq 0$

(*v*) $f(x)=\dfrac{\sin x-\cos x}{\frac{\pi}{4}-x}$ for $x\neq\dfrac{\pi}{4}$ (*vi*) $f(x)=\dfrac{\tan x}{x}$ for $x\neq 0$

(*vii*) $f(x)=\dfrac{e^x-e^{-x}}{x}$ for $x\neq 0$

(*viii*) $f(x)=\dfrac{x-2}{\sqrt{2x+1}-\sqrt{x+3}}$, $x\neq 2$

4. Examine the following functions for continuity in the intervals indicated.

(*i*) $f(x)=\dfrac{x^2+1}{(x-1)(x-3)}$ in $[-1, 2]$

(*ii*) $f(x)=\dfrac{x+1}{x^2-5x+6}$ in $[0, 2{\cdot}5]$

(*iii*) $f(x)=\dfrac{x^2+3x+2}{x^2+x+1}$ in $[0, 1]$

(*iv*) $f(x)=\dfrac{x^2+x+1}{2x^2-x-3}$ in $[0, 2]$

(*v*) $f(x)=\dfrac{x+1}{2x^2-x-1}$ in $[-1, 2]$

5. Show that the following functions are continuous every where.

(*i*) $f(x)=x^2+3x+1$; (*ii*) $f(x)=x\sin x+\cos x$.
(*iii*) $f(x)=e^x\sin x$ (*iv*) $f(x)=a^x\log x$ (*v*) $f(x)=2^x\sin 2x$
(*vi*) $f(x)=xe^x$ (*vii*) $f(x)=x^2e^{-x^2}+3^x\sin x$.

6. Draw the graphs of the following functions. Hence or otherwise find their points of discontinuity if any.

(*i*) $f(x)=2x+1$ $-1\leqslant x\leqslant 0$ (*vi*) $f(x)=3x+1$ $-1\leqslant x\leqslant 0$
$=x-2$ $0<x<2$ $=1+2x$ $0\leqslant x<3$
(*ii*) $f(x)=|x|$ $-2\leqslant x\leqslant 2$ (*vii*) $f(x)=2x+5$ $-2\leqslant x\leqslant -1$
(*iii*) $f(x)=|x-3|$ $0\leqslant x\leqslant 6$ $=x+0-1<x\leqslant 3$

(*iv*) $f(x)=x+3 \quad 0\leqslant x\leqslant 1$
$=4-2x \quad 1<x\leqslant 4$

(*v*) $f(x)=4+2x \quad -1\leqslant x\leqslant 2$
$=12-x \quad 2<x\leqslant 5$

(*viii*) $f(x)=x+1 \quad -1\leqslant x\leqslant 0$
$=2-x \quad 0<x\leqslant 2$
$=x-1 \quad 2<x\leqslant 4$

(*ix*) $f(x)=1-x \quad -1\leqslant x\leqslant 1$
$=2 \quad 1<x\leqslant 3$
$=1+x \quad 3<x\leqslant 5$

Prepare a table of values of the function

$$f(x)=\frac{1}{x-2} \quad \text{for} \quad x\neq 2,$$

in the neighbourhood of $x=2$. Hence, comment on its behaviour in the neighbourhood of 2.

ANSWERS

1. (*i*) Continuous ; (*ii*) discontinuous ; (*iii*) continuous (*iv*) continuous, (*v*) continuous.

3. (*i*) $f(3/2)=6$; (*ii*) $f(-1)=3$; (*iii*) $f(0)=0$ (*iv*) $f(0)=\log a$ (*v*) $f(\pi/4)=-\sqrt{2}$; (*vi*) $f(0)=1$; (*vii*) $f(c)=2$; (*viii*) $f(2)=2\sqrt{5}$.

4. Discontinuous (*i*) $x=1$; (*ii*) $x=2$, (*iv*) $x=3/2$; (*v*) $x=-1/2, x=1$

6. (*i*) Discontinuous at $x=0$; (*ii*) Continuous
(*iii*) Continuous (*iv*) Discontinuous at $x=1$
(*v*) Continuous at $x=2$ (*vi*) Continuous
(*vii*) Continuous (*viii*) Discontinous at $x=0, x=2$
(*ix*) Discontinuous at $x=1, x=3$.

3

Differentiation

You have studied derivatives of functions and some of their applications in the 11th standard. The functions considered were simple algebraic and trigonometric functions. We shall now consider other types of functions such as exponential, logarithmic and inverse functions. Some more applications of the derivatives will also be discussed. Let us first revise what you learnt last year.

3.1. Derivative

The derivative of a function f at a point a of its domain is defined as,

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

In general the derivative at x is.

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h(-f(x)}{h}}$$

'h' is called an elementary increment in x and $[f(x)+h)—f(x)]$ is then the corresponding increment in $f(x)$ or the value of the function.

If we write

$$y = f(x)$$

and denote an increment in x by δx. Then the corresponding increment in y is denoted by δy and we then have,

$$y + \delta y = f(x + \delta x)$$

$$\therefore \quad f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{y+\delta y-y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ is denoted by $\frac{dy}{dx}$. Thus we obtain another notation for the derivative *e.g.*

$f'(x) = \frac{dy}{dx}$, where $y = f(x)$

Similarly if u, v are functions of x their derivatives are $\frac{du}{dx}$ and $\frac{dv}{dx}$.

We may also write,

$$f'(x) = \frac{d}{dx}\left\{ f(x) \right\}$$

$$g'(x) = \frac{d}{dx}\left\{ g(x) \right\} \text{ etc.}$$

Note that $\frac{d}{dx}$ is treated as an operator which when applied to a function yields its derivative.

Illustrative Examples

Ex. 1. Use the definition to find the derivatives of the following functions.

(*i*) $\sqrt{x^2+2}$; (*ii*) $\frac{1}{x+3}$; (*iii*) $2x^2+3x$.

Sol. (*i*) $f(x) = (x^2+2)^{1/2}$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2+2]^{1/2}-[x^2+2]^{1/2}}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2+2]^{1/2}-[x^2+2]^{1/2}}{h} \times \times \frac{[(x+h)^2+2]^{1/2}+[x^2+2]^{1/2}}{[(x+h)^2+2]^{1/2}+[x^2+2]^{1/2}}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2+2]-[x^2+2]}{h\{[(x+h)^2+2]^{1/2}+[x^2+2]^{1/2}\}}$$

$$= \lim_{h \to 0} \frac{[x^2+2hx+h^2+2-x^2-2]}{h\{[(x+h)^2+2]^{1/2}+[x^2+2^{1/2}]\}}$$

$$= \lim_{h \to 0} \frac{h\,(2x+h)}{h\{[(x+h)^2+2]^{1/2}+[x^2+2]^{1/2}\}}.$$

$$= \lim_{h\to 0} \frac{h\,(2x+h)}{\{[(x+h)^2+2]^{1/2}+[x^2+2]^{1/2}}$$

$$= \lim_{h\to 0} \frac{2x+h}{\{[(x+h)^2+2]^{1/2}+[x^2+2]^{1/2}\}} \quad \because\ h\to 0,\ h\neq 0$$

$$= \frac{2x}{2[x^2+2]^{1/2}}$$

$$= \frac{x}{(x^2+2)^{1/2}}$$

(*ii*) $f(x) = \dfrac{1}{x+3}$

$$\therefore \quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{x+h+3} - \frac{1}{x+3}\right]$$

$$= \lim_{h\to 0} \frac{x+3-x-h-3}{h\,(x+3)\,(x+h+3)}$$

$$= \lim_{h\to 0} \frac{-h}{h\,(x+3)\,(x+h+3)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+3)\,(x+h+3} \quad \because\ h\to 0,\ h\neq 0.$$

$$= \frac{-1}{(x+3)^2}$$

(*iii*) $f(x) = 2x^2+3x$

$$\therefore \quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2\,(x+h)^2+3\,(x+h)-(2\,x^2+3x)}{h}$$

$$= \lim_{h\to 9} \frac{2x^2+4hx+2h^2+3x+3h-2x^2-3x}{h}$$

$$= \lim_{h\to 0} \frac{h\,(4x+2h+3)}{h}$$

$$= \lim_{h\to 0} (4x+2h+3) \quad \because\ h\to 0,\ h\neq 0.$$

$$= 4x+3.$$

Ex. 2. Find from from the first principles the derivatives of (*i*) $\cos^2 x$ (*ii*) $\tan x$.

Sol. (*i*) $f(x) = \cos^2 x$

$$f'(x) = \lim_{h\to 0} \frac{\cos^2 (x+a)-\cos^2 x}{h}$$

$$= \lim_{h\to 0} \frac{[\cos(x+h)] - \cos x][\cos(x+h) + \cos x]}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin\left((x+\frac{h}{2}\right)\sin\frac{h}{2}\; 2\cos\left(x+\frac{h}{2}\right)\cos\frac{h}{2}}{h}$$

$$= \lim_{h\to 0} \left[-2\sin\left(x+\frac{h}{2}\right)\right]\left[\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]$$

$$\left[\cos\left(x+\frac{h}{2}\right)\right]\left[\cos\frac{h}{2}\right]$$

$$= -2\sin x \cdot 1 \cdot \cos x \cdot 1$$

$$= -2\sin x\cos x$$

$$= -\sin 2x.$$

(*ii*) $f(x) = \tan x$

$$\therefore \quad f'(x) = \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to a} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \cdot \frac{1}{\cos(x+h)} \cdot \frac{1}{\cos x}$$

$$= 1 \cdot \frac{1}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \sec^2 x.$$

Ex. 3. Use the definition to find the derivative of $\sec x$.

Sol. $f(x) = \sec x$

$$\therefore \quad f(x) = \lim_{h\to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\frac{\cos x - \cos(x+h)}{\cos x \cdot \cos(x+h)}\right\}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{2 \sin\left(x+\frac{h}{2}\right) \cdot \sin\frac{h}{2}}{\cos(x+h) \cdot \cos x}$$

$$= \lim_{h\to 0} \left[\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]\left[\frac{\sin\left(x+\frac{h}{2}\right)}{\cos(x+h)}\right]\left[\frac{1}{\cos x}\right]$$

$$= 1 . \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} .$$

$$= \tan x \sec x.$$

3·2. Algebra of derivatives

Last year you learnt the rules of derivatives. We prove them again as they will be used frequently in further discussion.

Rules of differentiation

Rule 1. If u and v are functions of x and $y = u+v$, then

$$\boxed{\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}}$$

Proof. Let δy, δu and δv be the increments in y, u and v respectively corresponding to an increment δx in x, then

as x changes to $x+\delta x$,

u changes to $u+\delta u$

v changes to $v+\delta v$

and y changes to $y+\delta y$.

We assume that, as $\delta x \to 0$, δu, δv and $\delta y \to 0$.

Now, $y = u+v$

$\therefore$ $y+\delta y = (u+\delta u)+(v+\delta v)$

$\therefore$ $\delta y = (y+\delta y)-y$

$= (u+\delta u)+(v+\delta v)-(u+v)$

$= \delta u+\delta v.$

$\therefore$ $\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}$

$\therefore$ $\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}\right)$

$= \lim_{\delta x\to 0} \frac{\delta u}{\delta x} + \lim_{\delta x\to 0} \frac{\delta v}{\delta x}$

$= \frac{du}{dx} + \frac{dv}{dx}$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Cor. If $y = u+v+w$ then,

$$\frac{dy}{dx} = \frac{d}{dx}(u+v) + \frac{dw}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$$

Rule II. If u and v are functions of x and $y = u-v$, then,

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

The proof is exactly similar to that of Rule I and left as an exercise to the students.

Rule III. If u and v are functions of x and $y = u.v$, then,

$$\boxed{\frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}}$$

Proof. Let δy, δu and δv be the increments in y, u and v respectively corresponding to an increment δx in x. Then,

as x changes to $x+\delta x$,

u changes to $u+\delta u$

v changes to $v+\delta v$

and y changes to $y+\delta y$

We assume that as $\delta x \to 0$, δu, δv and $\delta y \to 0$

Now $y = u \cdot v$.

$$\therefore \quad y+\delta y = (u+\delta u)(v+\delta v)$$

$$\therefore \quad \delta y = (y+\delta y)-y = (u+\delta u)(v+\delta v)-uv$$

$$= uv+u\delta v+v\delta u+\delta u \cdot \delta v-uv.$$

$$= u\delta v+v\delta u+\delta u \cdot \delta v.$$

$$\therefore \quad \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v.\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x} \cdot \delta v.$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \left(u \cdot \frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\delta v \right)$$

$$= \lim_{\delta x \to 0} u \cdot \frac{\delta v}{\delta x} + \lim_{\delta x \to 0} v.\frac{\delta u}{\delta x} + \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \cdot \delta v$$

$$= u\left(\lim_{\delta x \to 0} \frac{\delta v}{\delta x} \right) + v\left(\lim_{\delta x \to 0} \frac{\delta u}{\delta x} \right) + \left(\lim_{\delta x \to 0} \frac{\delta u}{\delta x} \right) \times \left(\lim_{\delta x \to 0} \delta v \right)$$

$$= u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} + \frac{du}{dx} \times 0$$

$\because \quad \delta v \to 0$ as $\delta x \to 0$

$$\therefore \quad \frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \frac{du}{dx}$$

Cor. 1. If $y = ku$, where k is a constant then,

$$\boxed{\frac{dy}{dx} = k \frac{du}{dx}}$$

Proof. $y = ku$

$$\frac{dy}{dx} = k \frac{du}{dx} + u \frac{dk}{dx}$$

Since k is a constant, $\frac{dk}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = k \frac{du}{dx} + u \times 0 = k \cdot \frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = k \cdot \frac{du}{dx}$$

Cor. 2. If $y = uvw$, then,

$$\frac{dy}{dx} = uv \frac{dw}{dx} + uw \frac{dv}{dx} + vw \frac{du}{dx}$$

Proof. $y = uvw$.

$$\therefore \quad \frac{dy}{dx} = (uv) \frac{dw}{dx} + w \frac{d}{dx} (uv)$$

$$= uv \frac{dw}{dx} + w \left(u \frac{dv}{dx} + v \frac{du}{dx} \right)$$

$$\therefore \quad \frac{dy}{dx} = uv \frac{dw}{dx} + uw \frac{dv}{dx} + vw \frac{du}{dx}$$

In this way the result can be easily extended to the product of more than three functions.

Ex. 1. If $y = (x^2+x+1)(2x^2+3)$ find $\frac{dy}{dx}$

Sol. Here, $y = uv$, where

$u = x^2+x+1$ and $v = 2x^2+3$

$$\therefore \quad \frac{du}{dx} = 2x+1 \text{ and } \frac{dv}{dx} = 4x+0$$

$$\therefore \quad \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$= (x^2+x+1)\, 4x + (2x^2+3)\, (2x+1)$$

Ex. 2. If $y = 3x^2 \sin x$, find $\dfrac{dy}{dx}$.

Sol. $y = 3x^2 \sin x$

$$\therefore \quad \frac{dy}{dx} = 3x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(3x^2)$$

$$= 3x^2 \cos x + (\sin x)\, 6x$$

With a little practice, the student can work **mentally, and** write down the derivatives directly.

Rule IV. If u and v are functions of x and $y = \dfrac{u}{v}$; $v \neq 0$ then

$$\boxed{\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}}$$

Proof. Let δy, δu and δv be the increments in y, u and v respectively corresponding to an increment δx in x.

We assume that,

as $\quad \delta x \to 0$; $\delta y \to 0$, $\delta u \to 0$ and $\delta v \to 0$

We have $\quad y = \dfrac{u}{v} \quad \therefore \; y + \delta y = \dfrac{u + \delta u}{v + \delta v}$

$$\therefore \quad \delta y = \frac{u + \delta u}{v + dv} - \frac{u}{v}$$

$$= \frac{v(u + \delta u) - u(v + \delta v)}{v(v + \delta v)}$$

$$= \frac{vu + v\delta u - uv - u\delta v}{v(v + \delta v)}$$

$$= \frac{v\delta u - u\delta v}{v(v + \delta v)}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v + \delta v)}$$

[Note that on the right hand side, we have **divided the terms** in the numerator by δx.]

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v + \delta v)}$$

$$= \frac{\lim\limits_{\delta x \to 0}\left(v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}\right)}{\lim\limits_{\delta x \to 0} v(v+\delta v)}$$

$$= \frac{v\lim\limits_{\delta x \to 0}\frac{\delta u}{\delta x} - u\lim\limits_{\delta x \to 0}\frac{\delta v}{\delta x}}{v\lim\limits_{\delta x \to 0}(v+\delta v)}$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \because \quad \delta v \to 0, \text{ as } \delta x \to 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Ex. 1. If $y = \frac{x^2-1}{x^2+1}$ find $\frac{dy}{dx}$.

Sol. Here $y = \frac{u}{v}$ where $u = x^2 - 1$ and $v = x^2 + 1$

$$\therefore \quad \frac{du}{dx} = 2x \text{ and } \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

Ex. 2. If $y = \frac{\sin x}{1+\cos x}$ find $\frac{dy}{dx}$.

Sol. Here $y = \frac{u}{v}$ where $u = \sin x$ and $v = 1 + \cos x$.

$$\therefore \quad \frac{du}{dx} = \cos x \text{ and } \frac{du}{dx} = -\sin x$$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{1+\cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{2}{(1+\cos x)^2} \qquad \because \quad \sin^2 x + \cos^2 x = 1$$

Ex. 3. If $y = \tan x$ find $\frac{dy}{dx}$.

Sol. $$y = \frac{\sin x}{\cos x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x . \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x \quad \therefore \text{ If } y = \tan x, \frac{dy}{dx} = \sec^2 x$$

Ex. 4 If $y = x \sin x + \cos x$, find dy/dx.

Sol. $$\frac{dy}{dx} = x.\frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x) + \frac{d}{dx}(\cos x)$$

$$= x \cos x + \sin x - \sin x$$

$$= x \cos x.$$

3·3. Consider the function $y = \cos^5 x$.

This is not a standard form discussed earlier, whose derivative is known nor can it be expressed as the sum, difference etc. of standard forms. We therefore break it into a number of standard forms to facilitate writing its derivative. Thus we write

$$y = u^5 \qquad \text{where} \qquad u = \cos x$$

Now y as a function of u and u as a function of x are standard forms. Thus y is expressed as a function of u in a standard form where u is a standard function of x. In this case we call y as a composite function of x.

Similarly if $y = \tan(\sqrt{x^2+1})$, we can express this composite function in terms of simple standard forms as follows.

$$y = \tan u \text{ where } u = \sqrt{v}$$

where $$v = x^2 + 1$$

Rule V. If y is a function of u and u is a function of x then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Proof. We assume that the derivatives $\frac{dy}{dx}$, $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.

Let x change to $x+\delta x$. Correspondingly let u change to $u+\delta u$ and consequently y change to $y+\delta y$. Then because $\frac{dy}{dx}$, $\frac{dy}{du}$ and $\frac{du}{dx}$ exist, as $\delta x \to 0$, $\delta y \to 0$ and $\delta u \to 0$

Now $$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$$

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right)$$

$$= \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u} \right) \cdot \lim_{\delta x \to 0} \left(\frac{\delta u}{dx} \right)$$

$$= \lim_{\delta u \to 0} \left(\frac{\delta y}{\delta x} \right) \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} \right)$$

$\because$ as $\delta x \to 0$, $\delta u \to 0$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Cor. If $y = u^n$, where u is a function of x then

$$\frac{dy}{dx} = nu^{n-1} \frac{du}{dx}$$

Proof. $y = u^n \therefore \frac{dy}{du} = nu^{n-1}$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = nu^{n-1} \cdot \frac{du}{dx}$$

Ex. 1. If $y = \sin(1+x^2)$ find $\frac{dy}{dx}$.

Sol. Write $u = 1+x^2$.

$$\therefore \quad \frac{du}{dx} = 2x.$$

Now $y = \sin u$

$$\therefore \quad \frac{dy}{du} = \cos u$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u)\ 2x$$

$$= 2x \cos(1+x^2)$$

Ex. 2. If $y=(x^3+3x-5)^{7/2}$, find $\frac{dy}{dx}$.

Sol. Write $u=x^2+3x-5$

$\therefore$ $y=u^{7/2}$ and $u=x^2+3x-5$

$\therefore$ $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{7}{2}u^{5/2}(2x+3)$

$=\frac{7}{2}(x^2+3x-5)^{5/2}(2x+3)$

Ex. 3. If $y=\sqrt{2+4x+x^2}$, find $\frac{dy}{dx}$.

Sol. Write $u=2+4x+x^2$

$\therefore$ $y=u^{1/2}$ and $u=2+4x+x^2$

$\therefore$ $\frac{dy}{dx}=\frac{1}{2u^{1/2}}(4+2x)=\frac{1}{u^{1/2}}(2+x)$

$=\frac{x+2}{\sqrt{x^2+4x+2}}$.

Ex. 4. If $y=\frac{x}{\sqrt{1+x^2}}$, find $\frac{dy}{dx}$.

Sol. $y=\frac{x}{(1+x^2)^{1/2}}$

$\therefore$ $\frac{dy}{dx}=\frac{(1+x^2)^{1/2}\frac{d}{dx}(x)-x\frac{d}{dx}(1+x^2)^{1/2}}{(1+x^2)}$

$=\frac{(1+x^2)^{1/2}-\frac{x\,.\,2x}{2(1+x^2)^{1/2}}}{(1+x^2)}$

$=\frac{1+x^2-x^2}{(1+x^2)(1+x^2)^{1/2}}$

$=\frac{1}{(1+x^2)^{3/2}}$

Note. $v=(1+x^2)^{1/2}$ *i. e.* $v=u^{1/2}$ where $u=(1+x)^2$

$\therefore$ $\frac{dv}{dx}=\frac{1}{2}u^{-1/2}.\ 2x=\frac{1}{2}(1+x^2)^{-1/2}.\ 2x$

In general, if $y=\sqrt{u}$ then $\frac{dy}{dx}=\frac{1}{2\sqrt{u}}\frac{du}{dx}$

[The student should work out the steps mentally.]

Ex. 5. If $y=\sin 2x+\cos 2x$, find $\frac{dy}{dx}$.

Sol. [We now omit some steps which the student can easily visualise and write the result directly.]

$$\frac{dy}{dx}=2\cos 2x-2\sin 2x.$$

Ex. 6. If $y=\frac{\sin x}{(1+x^2)^{1/2}}$.

$$\therefore \quad \frac{dy}{dx}=\frac{(1+x^2)^{1/2}\cos x-\sin x\ \frac{1}{2}(1+x^2)^{-1/2}\,2x}{[(1+x^2)^{1.2}]^2}$$

$$=\frac{(1+x^2)^{1/2}\cos x-\dfrac{x\sin x}{(1+x^2)^{1/2}}}{(1+x^2)}$$

$$=\frac{(1+x^2)\cos x-x\sin x}{(1+x^2)^{3/2}}$$

Ex. 7. If $y=x^3\tan x^2$, find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx}=x^3\ \frac{d}{dx}(\tan x^2)+\tan x^2\ \frac{d}{dx}(x^3)$

$$=x^3.(\sec x^2)\,2x+\tan x^2.(3x^2)$$

$$=2x^4\sec x^2-3x^2\tan x^2$$

$$=x^2(2x^2\sec x^2+3\tan x^2).$$

3·4. Standard Forms.

To obtain the derivatives of functions we use the above rules and some standard derivatives. We state below the standard derivatives obtained last year and derive some more. The student should prove the results obtained last year.

(*i*) If $y=k$ then $\frac{dy}{dx}=0$.

(*ii*) If $y=x^n$ then $\frac{dy}{dx}=nx^{n-1}$, n being any real number.

(*iii*) If $y=\sin x$ than $\frac{dy}{dx}=\cos x$.

(*iv*) If $y=\cos x$, then $\frac{dy}{dx}=-\sin x$.

(*v*) If $y=\tan x$, then $\frac{dy}{dx}=\sec^2 x$.

The above 5 derivatives were obtained last year-We now consider some more functions.

(*vi*) $y=\operatorname{cosec} x$

$\therefore \quad y=\dfrac{1}{\sin x}$

$\therefore \quad \dfrac{dy}{dx}=-\dfrac{1}{\sin^2 x}\cdot \cos x$

$=-\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x}=-\operatorname{cosec} x.\ \cot x.$

$\therefore$ If $y=\operatorname{cosec} x$, then $\dfrac{dy}{dx}=-\operatorname{cosec} x.\ \cot x.$

(*vii*) $y=\sec x$

$\therefore \quad y=\dfrac{1}{\cos x}$

$\therefore \quad \dfrac{dy}{dx}=-\dfrac{1}{\cos^2 x}\cdot(-\sin x)$

$=\dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x}=\sec x.\ \tan x$

$\therefore$ If $y=\sec x$, then $\dfrac{dy}{dx}=\sec x.\ \tan x.$

(*viii*) $y=\cot x$

$\therefore \quad y=\dfrac{\cos x}{\sin x}$

$\therefore \quad \dfrac{dy}{dx}=\dfrac{\sin x\,(-\sin x)-\cos x\,(\cos x)}{\sin^2 x}$

$=\dfrac{-[\sin^2 x+\cos^2 x]}{\sin^2 x}$

$=-\dfrac{1}{\sin^2 x} \quad \because \ \sin^2 x+\cos^2 x=1$

$=-\operatorname{cosec}^2 x$

$\therefore$ If $y=\cot x$, $\dfrac{dy}{dx}=-\operatorname{cosec}^2 x$

(*ix*) $y=a^x$

$\dfrac{dy}{dx} \quad \lim\limits_{\delta x\to 0}\dfrac{a^{x+\delta x}-a^x}{\delta x}$

$=\lim\limits_{\delta x\to 0}\dfrac{a^x\,(a^{\delta x}-1)}{\delta x}$

$=a^x\lim\limits_{\delta x\to 0}\dfrac{a^{\delta x}-1}{\delta x}$

$$= a^x \log_e a \quad \because \lim_{h\to 0} \frac{a^h - 1}{h} = \log_e a$$

$\therefore$ if $y = a^x$, $\dfrac{dy}{dx} = a^x \log a$

(x) $y = e^x$

$$\therefore \frac{dy}{dx} = a^x \log_e e$$

$$= e^x \qquad \because \log_e e = 1$$

$\therefore$ if $y = e^x$, $\dfrac{dy}{dx} = e^x$

(xi) $y = \log_e x$

$$\therefore \frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\log_e (x+\delta x) - \log_e x}{\delta x}$$

$$= \lim_{\delta x\to 0} \frac{1}{\delta x} \log_e \frac{x+\delta x}{x}$$

$$= \lim_{\delta x\to 0} \frac{1}{x} \cdot \frac{x}{\delta x} \cdot \log_e \left[1 + \frac{\delta x}{x}\right]$$

$$= \lim_{\delta x\to 0} \frac{1}{x} \log_e \left[1 + \frac{\delta x}{x}\right]^{x/\delta x}$$

Now put $u = \dfrac{\delta x}{x}$ $\quad \therefore \quad u \to 0$ as $\delta x \to 0$

$$\therefore \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x\to 0} \log_e [1+u]^{1/u}$$

$$= \frac{1}{x} \log_e \left[\lim_{u\to 0} (1+u)^{1/u}\right]$$

$$= \frac{1}{x} \log_e e \qquad \because \lim_{u\to 0} [1+u]^{1/u} = e$$

$$= \frac{1}{x} \qquad \because \log_e e = 1$$

$\therefore$ if $y = \log_e x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Illustrative Examples

Ex. 1. Find the derivatives of the following functions.

(i) $y = x^2 \sin 2x$;

(ii) $y = \dfrac{x \sin x}{1+\cos x}$, $x \neq (2n+1)\pi$

(*iii*) $y=\dfrac{\sin x+\cos x}{\sin x-\cos x}$, $\sin x\neq\cos x$

Sol. (*i*) $y=x^2 \sin 2x$

$$\therefore \quad \frac{dy}{dx}=x^2 \cos 2x\,(2)+2x \sin 2x$$

$$=2x\,(x \cos 2x+\sin 2x)$$

(*ii*) $y=\dfrac{x \sin x}{1+\cos x}$, $x\neq(2n+1)\pi$

$$\therefore \quad \frac{dy}{dx}=\frac{(1+\cos x)(\sin x+x\cos x)-x \sin x(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x(1+\cos x)+x \cos x+x \cos^2 x+x \sin^2 x)}{(1+\cos x)^2}$$

$$=\frac{\sin x\,(1+\cos x)+x \cos x+x\,(\cos^2 x+\sin^2 x)}{(1+\cos x)^2}$$

$$=\frac{\sin x\,(1+\cos x)+x\,(\cos x+1)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)\,(\sin x+x)}{(1+\cos x)^2}$$

$$=\frac{\sin x+x}{1+\cos x}.$$

(*iii*) $y=\dfrac{\sin x+\cos x}{\sin x-\cos x}$

$$\therefore \quad \frac{dy}{dx}=\frac{(\sin x-\cos x)\,(\cos x-\sin x)-(\sin x+\cos x)}{(\sin x-\cos x)^2}\times(\cos x+\sin x)$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$=\frac{-\sin^2 x+2 \sin x \cos x+-\cos^2 x-\sin^2 x-2 \sin x \cos x-\cos^2 x}{(\sin x-\cos x)^2}$$

$$=\frac{-2\,(\sin^2 x+\cos^2 x)}{(\sin x-\cos x)^2}=\frac{-2}{(\sin x-\cos x)^2}.$$

Ex. 2. Find the derivatives of the following functions.

(*i*) $\sec x^2$ (*ii*) $\dfrac{\sec x+\tan x}{\sec x-\tan x}$

(*iii*) $\sqrt{\cot x}$, $0<x\leqslant\pi$

Sol. (*i*) $y=\sec x^2$

Put $x^2=u$ $\quad\therefore\quad \dfrac{du}{dx}=2x$

$\therefore \quad y = \sec u$ and $u = x^2$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$= \sec u \,.\, \tan u \,(2x)$

$= 2x \sec x^2 \cdot \tan x^2$

(*ii*) $\quad y = \dfrac{\sec x + \tan x}{\sec x - \tan x}$

$$\therefore \quad \frac{dy}{dx} = \frac{(\sec x - \tan x)(\sec x \tan x + \sec^2 x) - (\sec x + \tan x)(\sec x \tan x - \sec^2 x)}{(\sec x - \tan x)^2}$$

$$= \frac{\sec x (\sec x - \tan x)(\sec x + \tan x) - \sec x (\sec x + \tan x) \times (\tan x - \sec x)}{(\sec x - \tan x)^2}$$

$$= \frac{\sec x (\sec^2 x - \tan^2 x) - \sec x (\tan^2 x - \sec^2 x)}{(\sec x - \tan x)^2}$$

$$= \frac{\sec x (1) - \sec x (-1)}{(\sec x - \tan x)^2} \qquad \because \quad \sec^2 x - \tan^2 x = 1$$

$$= \frac{2 \sec x}{(\sec x - \tan x)^2}.$$

(*iii*) $\quad y = \sqrt{\cot x}$

$= (\cot x)^{1/2}$

Put $y = u \cot x \qquad \therefore \quad \dfrac{du}{dx} = -\operatorname{cosec}^2 x$

$\therefore \quad y = u^{1/2}$ and $u = \cot x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$= \dfrac{1}{2} u^{-1/2} \dfrac{du}{dx}$

$= \dfrac{1}{2} (\cot x)^{-1/2} (-\operatorname{cosec}^2 x)$

$= -\dfrac{\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$

Ex. 3. Find $\dfrac{dy}{dx}$ in the following cases.

(*i*) $\quad y = 2^x \qquad$ (*ii*) $\quad y = x \,.\, 3^x \qquad$ (*iii*) $\quad y = a^{\sin ?x}$

Sol. (*i*) $y=2^x$

$\therefore \quad \frac{dy}{dx}=2^x \log_e 2$

(*ii*) $y=x.\ 3^x$

$\therefore \quad \frac{dy}{dx}=x.\ 3^x \log 3+3^x$

$=3^x (x \log 3+1)$

$=3^x (\log 3^x+1)$

(*iii*) $y=a^{\sin 2x}$

Put $\sin 2x=u \qquad \therefore \quad \frac{du}{dx}=2 \cos 2x$

$\therefore \quad y=a^u$ and $u=\sin 2x$

$\therefore \quad \frac{dy}{dx}=\frac{dy}{du} \cdot \frac{du}{dx}$

$=a^u \log a.\ 2 \cos 2x.$

$=2 \cos 2x \log a.\ a^u.$

$=2 \cos 2x \log a.\ a.^{\sin 2x}$

Remark. If $y=a^{f(x)}$ then put $u=f(x)$

$\therefore \quad y=a^u$ and $u=f(x)$

$\therefore \quad \frac{dy}{dx}=\frac{dy}{du} \cdot \frac{du}{dx}$

$=a^u \log a.\ f'(x)$

$=a^{f(x)} f'(x).\ \log a.$

Ex. 4. Find the derivatives of

(*i*) $y=\log (\sin x+\cos x)$,

(*ii*) $y=\log [x+\sqrt{1+x^2}]$

Sol. (*i*) $y=\log (\sin x+\cos x)$

put $u=\sin x+\cos x \qquad \frac{du}{dx}=\cos x-\sin x$

$\therefore \quad y=\log u$ and $u=\sin x+\cos x$

$\therefore \quad \frac{dy}{dx}=\frac{dy}{du} \cdot \frac{du}{dx}$

$=\frac{1}{u} (\cos x-\sin x)$

$=\frac{\cos x-\sin x}{\cos x+\sin x}$

(*ii*) $y=\log [x+\sqrt{1+x^2}]$

Put $u=x+\sqrt{1+x^2}$

$$\therefore \frac{du}{dx}=1+\frac{x}{\sqrt{1+x^2}}$$

$$=\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$\therefore$ $y=\log u$ and $u=x+\sqrt{1+x^2}$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\frac{1}{u}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$=\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$=\frac{1}{\sqrt{1+x^2}}\cdot$$

Remark. If $y=\log f(x)$, put $u=f(x)$

Then $y=\log u$ and $u=f(x)$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\frac{1}{u}f'(x)$$

$$=\frac{f'(x)}{f(x)}$$

$\therefore$ if $y=\log f(x)$, $\frac{dy}{dx}=\frac{f'(x)}{f(x)}$

Ex. 5. Find $\frac{dy}{dx}$ if, (*i*) $y=\log \sin x$, (*ii*) $y=5^x.\ x^5$ (*iii*) $y=\log_{10}(1+x^2)$ (*iv*) $y=e^{2x}\sin 3x$.

Sol. (*i*) $y=\log \sin x$.

$$\therefore \frac{dy}{dx}=\frac{1}{\sin x}\cos x$$

$$=\cot x$$

(*ii*) $y=5^x.\ x^5$

$$\therefore \frac{dy}{dx}=5^x.(\log 5).x^5+5^x\ 5x^4$$

$$=5^x(x^5\log^5+5x^4)$$

(*iii*) $y=\log_{10}(1+x^2)$

$\therefore$ Changing the base to e.

$$y=\frac{\log(1+x^2)}{\log_e 10}$$

$$=\log_{10}e \cdot \log_e(1+x^2)$$

$$\therefore \quad \frac{dy}{dx}=\log_{10}e \times \frac{2x}{1+x^2}$$

$$=\frac{2x\log_{10}e}{1+x^2}$$

(*iv*) $y=e^{2x}\sin 3x$

$$\frac{dy}{dx}=2e^{2x}\sin 3x+3e^{2x}\cos 3x$$

$$=e^{2x}(2\sin 3x+3\cos 3x)$$

Ex. 6. If $f(x)=\log\{\log(\log x)\}$ find $f'(x)$

Sol. Write $u=\log x$, $v=\log(\log x)=\log u$.

$\therefore \quad f(x)=\log v$

$\therefore \quad f(x)=\frac{1}{v}\cdot\frac{dv}{dx}$

Now, $v=\log u$ where $u=\log x$

$$\therefore \quad \frac{dv}{dx}=\frac{1}{u}\cdot\frac{du}{dx}$$

$$=\frac{1}{u}\cdot\frac{1}{x}$$

$$\therefore \quad f'(x)=\frac{1}{v}\cdot\frac{1}{u}\cdot\frac{1}{x}$$

$$=\frac{1}{\log(\log x)}\cdot\frac{1}{\log x}\cdot\frac{1}{x}\cdot$$

$$=\frac{1}{x.\log x.\log(\log x)}$$

Ex. 7. If $y=\sin^2 x^\circ$ find $\frac{dy}{dx}$.

Sol. $y=\sin^2 x^\circ$

$$=\sin^2\left(\frac{\pi}{180}x\right)^c$$

$$\therefore \quad \frac{dy}{dx}=2\sin\left(\frac{\pi}{180}x\right)^c\cos\left(\frac{\pi}{180}x\right)^c\frac{\pi}{180}$$

$$= \frac{\pi}{90} \sin x^\circ \cos x^\circ$$

$$= \frac{\pi}{180} \sin (2x)^\circ$$

Ex. 8. Find $\frac{dy}{dx}$ if (*i*) $y = \log_a x$

(*ii*) $y = e^{(x \sec^2 x - \tan x)}$

Sol. (*i*) $y = \log_a x$

$$= \frac{\log_e x}{\log_e a}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\log_e a} \cdot \frac{1}{x}$$

$$= \frac{1}{x \cdot \log a}$$

(*iii*) $y = e^{(x \sec^2 x - \tan x)}$

Put $u = x \sec^2 x - \tan x$

$$\therefore \quad \frac{du}{dx} = x\, 2 \sec x . \tan x + \sec^2 x - \sec^2 x$$

$$= 2x \sec x \tan x$$

Now, $y = e^u$, and $u = x \sec^2 x - \tan x$

$$\therefore \quad \frac{dy}{dx} = e^u . \frac{du}{dx}$$

$$= e^{x \sec^2 x - \tan x} . 2x \sec x . \tan x$$

Remark. If $y = e^{f(x)}$ put $u = f(x)$

$$\therefore \quad \frac{du}{dx} = f'(x)$$

Now, $y = e^u$ and $u = f(x)$

$$\therefore \quad \frac{dy}{dx} = e^u \frac{du}{dx}$$

$$= e^{f(x)} f'(x)$$

Ex. 9. Find $\frac{dy}{dx}$ if (*i*) $y = \sqrt{x\sqrt{x}}$

(*ii*) $y = \sqrt{x + \sqrt{x}}$

Sol. (*i*) $y = \sqrt{x\sqrt{x}}$

$$= (x\sqrt{x})^{1/2}$$

$$= (x . x^{1/2})^{1/2}$$

$$= (x^{3/2})^{1/2}$$

$$\therefore \quad y = x^{3/4}$$

$$\therefore \quad \frac{dy}{dx} = \frac{3}{4} x^{3/4-1}$$

$$= \frac{3}{4} x^{-1/4}$$

(*ii*) $y = \sqrt{x+\sqrt{x}}$

$$= (x+x^{1/2})^{1/2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2} (x+x^{1/2})^{1/2-1} \left(1+\frac{1}{2} x^{-1/2} \right)$$

$$= \frac{1}{2} (x+x^{1/2})^{-1/2} \left(1+\frac{1}{2\sqrt{x}} \right)$$

$$= \frac{1}{2(x+x^{1/2})^{1/2}} \cdot \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

$$= \frac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}.$$

EXERCISE

1. Find the derivatives of the following functions using definition of the derivative—

   (*i*) $y = 2x^2 - x$, (*ii*) $y = \cos^2 x$

   (*iii*) $\frac{1}{x^2+1}$.

2. Find the derivatives of the following functions.

   (*i*) $(a+x)\sqrt{a-x}$ (*ii*) $\frac{1}{\sqrt{x+2}+\sqrt{x}}$,

   (*iii*) $\frac{1}{\sqrt{x^2+4}+x}$ (*iv*) $\sqrt{\frac{1+x}{1-x}}$

   (*v*) $\sqrt{x\sqrt{x\sqrt{x}}}$. (*vi*) $\sqrt{x+\sqrt{x+\sqrt{x}}}$

3. Find $\frac{dy}{dx}$ in the following cases,

   (*i*) $y = \sin x°$, (*ii*) $y = \cos^2 x°$,

   (*iii*) $y = \tan^2 x°$, (*iv*) $y = \sec x°$.

4. Find $\frac{dy}{dx}$ if,

   (*i*) $y = x^2 \tan x$ (*ii*) $y = (x+1)\cos(\lambda x+p)$

   (*iii*) $y = \sqrt{a \sin^2 x + b \cos^2 x}$, (*iv*) $y = \frac{\tan x - 1}{\sec x}$,

(v) $y=\log\,[x-\sqrt{1-x^2}\,]$, (vi) $y=\log\left[\frac{1+x}{1-x}\right]$,

(vii) $y=\sin\,(\cos x)$, (viii) $\sin\,(x+a)\,\sin\,(x-a)$

5. Differentiate w.r.t. x

(i) $\frac{\sin x}{1+\cos x}$, (ii) $\frac{\cos x-\sin x}{\cos x+\sin x}$,

(iii) $\frac{1+2\sin x}{2+\sin x}$, (iv) $\sqrt{\frac{1-\cos x}{1+\cos x}}$,

(v) $\frac{1+\tan x}{1-\tan x}$, (vi) $\frac{1}{3}\tan^3 x-\tan x+x$

(vii) $x\cos x-\sin x$, (viii) $(x^2+1)\sin x+2x\cos x$

(ix) $x\tan^2 x$, (x) $\sqrt{\cos 2x}$,

(xi) $\sqrt{\tan x}$, (xii) $\sqrt{\sec x+\tan x}$.

6. Differentiate *w.r.t.* x.

(i) 3^{x^2}, (ii) $5^x\log x$,

(iii) $e^{-x^2}\sin^2 x$ (iv) $e^{\sin x}\cdot\cos x$,

(v) $10^x\sin x$, (vi) $2^x.e^x$,

(vii) $a^{x\sin^2 x}$, (vii) $e^x\sin x\cos 2x$,

(ix) $2^{(\tan x-\cot x)}$, (x) $a^{\sin x}\cos x$.

7. Find the derivatives of

(i) $\log\,(\log x^2)$, (ii) $\log x^x$,

(iii) $\log\,(\sec x+\tan x)$, (iv) $\log\sqrt{\frac{1+\cos x}{1-\cos x}}$,

(v) $\log\,(\cos x+\sin x)$, (vi) $\log_2(\log_4 x)$,

(vii) $\log\,(x\sin x+\cos x)$, (viii) $\cos\,(\log x)$,

(ix) $\log\frac{1+\sqrt{x}}{1-\sqrt{x}}$.

8. Differentiate *w.r.t.* x.

(i) $\log\frac{\sqrt{x^2+1}-x}{\sqrt{x^2-1}+x}$, (ii) $\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$

(iii) $(\log x)^3$ (iv) $\sqrt{1+x^2}-\log\frac{x+\sqrt{1+x^2}}{x}$,

(v) $\sin\,(\log x)$, (vi) $\log\,(x^2-\sin x)$

(vii) $\log\sqrt{\frac{1+x}{1-x}}$, (viii) $-\frac{\cos x}{2\sin^2 x}+\frac{1}{2}\log\tan\frac{x}{2}$,

(ix) $\log\sqrt{x^2+x+1}$.

9. Find the derivatives of the following functions.

(*i*) $2^{\log \tan x}$, (*ii*) $3^{\sin x}$,

(*iii*) $e^{\log \tan x}$, (*iv*) $a^x \sin 2x$,

(*v*) $x^p e^{\sin x}$, (*vi*) $e^x \log \sin x$,

(*vii*) $x^{\log x}$, (*viii*) $x^{\sin x}$,

(*ix*) $e^{\sqrt{x}} \cos x$, (*x*) $\log \dfrac{e^x}{1+e^x}$,

(*xi*) a^{x^2} (*xii*) $5^{x+\frac{1}{x}}$.

ANSWERS

1. (*i*) $(4x-1)$, (*ii*) $-2 \cos x \,.\, \sin x$, (*iii*) $\dfrac{-2x}{(x^2+1)^2}$.

2. (*i*) $\dfrac{a-3x}{2\sqrt{a-x}}$, (*ii*) $\dfrac{1}{4}[(x+2)^{-\frac{1}{2}}-x^{-\frac{1}{2}}]$,

(*iii*) $\dfrac{1}{4}\left[\dfrac{x}{\sqrt{x^2+4}}-1\right]$ (*iv*) $\dfrac{1}{(1-x)(1-x^2)^{1/2}}$, (*v*) $\dfrac{7}{8}x^{-1/8}$,

(*iv*) $\dfrac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right)\right]$.

3. (*i*) $\dfrac{\pi}{180}\cos x^\circ$, (*ii*) $-\dfrac{\pi}{90}\cos x^\circ \sin x^\circ$,

(*iii*) $\dfrac{\pi}{90}\tan x^\circ \sec^2 x^\circ$ (*iv*) $\dfrac{\pi}{180}\sec x^\circ \tan x^\circ$

4. (*i*) $2x \tan x+x^2 \sec^2 x$, (*ii*) $\cos(\lambda x+p)-\lambda(x+1)\sin(\lambda x+p)$,

(*iii*) $\dfrac{(a-b)\sin x \cos x}{[a \sin^2 x+b \sin^2 x]^{1/2}}$, (*iv*) $\cos x+\sin x$,

(*v*) $\dfrac{x+\sqrt{1-x^2}}{\sqrt{1-x^2}\,[x-\sqrt{1-x^2}]}$, (*vi*) $\dfrac{2}{1-x^2}$, (*vii*) $-\cos(\cos x)\sin x$,

(*viii*) $\sin 2x$.

5. (*i*) $\sec^2 \dfrac{x}{2}$, (*ii*) $\sec^2\left(x-\dfrac{\pi}{4}\right)$ (*iii*) $\dfrac{3\cos x}{(2+\sin x)^2}$,

(*iv*) $\sec^2 \dfrac{x}{2}$, (*v*) $\sec^2\left(x+\dfrac{\pi}{4}\right)$, (*vi*) $\tan^4 x$, (*vii*) $-x \sin^2 x$

(*viii*) $(x^2+3)\cos x$, (*ix*) $\tan x(\tan x+x \tan^2 x+x)$, (*x*) $\dfrac{-\sin 2x}{\sqrt{\cos 2x}}$,

(*xi*) $\dfrac{\sec^2 x}{\sqrt{\tan x}}$, (*xii*) $\dfrac{1}{2}\sec x\sqrt{\sec x+\tan x}$.

6. (*i*) $3^{x^2} \cdot 2x \log 3$, (*ii*) $5^{x \log x} \cdot \log 5(\log x+1)$,

(*iii*) $e^{-x^2}(\cos x - 2x \sin x)$, (*iv*) $e^{\sin x}(\cos^2 x - \sin x)$,

(*v*) $10x^{\sin x} \log 10.(\sin x + x \cos x)$, (*vi*) $2^{xe^x} \log 2 \cdot e^x (x+1)$,

(*vii*) $a^x . \log a [\sin^2 x + 2x \sin x . \cos x]$,

(*viii*) $-\sqrt{2}\, e^x \sin 2x \sin\left(x + \frac{\pi}{4}\right)$,

(*ix*) $2^{\tan x - \cot x} . \log 2 . \sec^2 x \operatorname{cosec}^2 x$,

(*x*) $a^{\sin x}(\log a \cos^2 x - \sin x . \cos x)$.

7. (*i*) $\frac{2}{x \log (x^2)}$ (*ii*) $\log x + 1$

(*iii*) $\sec x$ (*iv*) $\frac{-2}{\sin x}$

(*v*) $\frac{\sin x - \cos x}{\sin x + \cos x}$ (*vi*) $\frac{1}{x \log 2}$

(*vii*) $\frac{x \cos x}{x \sin x + \cos x}$ (*viii*) $-\frac{1}{x} \sin$

(*ix*) $\frac{1}{\sqrt{x}(1-x)}$

8. (*i*) $-\left[\frac{1}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2+1}}\right]$ (*ii*) $\sec x$

(*iii*) $\frac{3(\log x)^2}{x}$ (*iv*) $\frac{(x-1)+\sqrt{1+x^2}}{\sqrt{1+x^2}}$

(*v*) $\frac{\cos(\log x)}{x}$ (*vi*) $\frac{2x - \cos x}{x^2 - \sin x}$

(*vii*) $\frac{1}{1-x^2}$ (*viii*) $\operatorname{cosec}^3 x$

(*ix*) $\frac{2x+1}{\sqrt{x^2+x+1}}$

9. (*i*) $2^{\log \tan x - 1} \log 2 . \operatorname{cosec} 2x$.

(*ii*) $3^{\sin x} \cos x . \log 3$.

(*iii*) $\frac{1}{2} e^{\log \tan x} \sec 2x$.

(*iv*) $a^x (\log a . \sin 2x + 2 \cos 2x)$

(*v*) $x^{p-1} e^{\sin x}(p + x \cos x)$

(*vi*) $e^x (\log \sin x + \cot x)$

(*vii*) $2x^{\log x - 1} . \log x$

(*viii*) $x^{\sin x}\left(\cos x \log x + \frac{\sin x}{x}\right)$

(*ix*) $e^{\sqrt{x}}\left[\frac{\cos x}{2\sqrt{x}}-\sin x\right]$ (*x*) $\frac{1}{1+e^x}$

(*xi*) $a^{x^2}\,2x\log a$ (*xiii*) $5^{x+\frac{1}{x}}\times\frac{x^2-1}{x}\log 5$.

3·5. Inverse Functions.

In the 11th standard, you have seen that if a function f from a set A on to another set B is on-to and one to one, then the inverse function exists.

Then if f^{-1} is the inverse of f, we have,

if $y=f(x)$, then $x=f^{-1}(y)$

We shall now prove that,

$f'(x).\{f^{-1}(y)\}'=1$

or in a simple form $\frac{dy}{dx}\cdot\frac{dx}{dy}=1$

Note that $\frac{dx}{dy}$ is the derivative of x *w.r.t.* y. The above relation implies that in this case *i.e.* when the inverse function exists $\frac{dx}{dy}$ is simply the reciprocal of $\frac{dy}{dx}$.

Proof :

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{1}{\frac{\delta x}{\delta y}}$$

$$=\lim_{\delta y\to 0}\frac{1}{\frac{\delta x}{\delta y}} \qquad \because \text{ if } \delta x\to 0, \text{ then } \delta y\to 0$$

$$=\frac{1}{\lim_{\delta y\to 0}\frac{\delta x}{\delta y}}$$

$$=\frac{1}{\frac{dx}{dy}}$$

$$\therefore \quad \frac{dy}{dx}\cdot\frac{dx}{dy}=1$$

Illustrative Examples

Ex. 1. If $x=\sqrt{1+y^2}$ and $x>0$, $x>0$, show that $\frac{dy}{dx}=\frac{x}{y}$.

Sol. $x=\sqrt{1+y}$

$\therefore$ $x^2=1+y^2$

and $y=\sqrt{x^2-1}$

$\therefore$ the domain of the function is $x\geqslant 1$, since $x>0$. The range is $y\geqslant 0$.

$\therefore$ the function (y as a function of x) is defined from $\{x/x\geqslant 1\}$ to $\{y/y\geqslant 0\}$. It is one to one and on to.

$\therefore$ the inverse exists and we have,

$$\frac{dy}{dx}\cdot\frac{dx}{dy}=1.$$

Now $x=\sqrt{1+y^2}$

$\therefore$ $\dfrac{dx}{dy}=\dfrac{1}{2}(1+y^2)^{-1/2}\cdot 2y$

$=\dfrac{y}{\sqrt{1+y^2}}$

$=\dfrac{y}{x}$ $\quad\because\quad x=\sqrt{1+y^2}$

$\therefore$ $\dfrac{dy}{dx}=\dfrac{1}{\dfrac{dx}{dy}}$

$=\dfrac{x}{y}$.

Ex. 2. If $y=\log_e x$, show that $\dfrac{dy}{dx}=\dfrac{1}{x}$.

Sol. You have seen last year that the logarithmic function is the inverse of the exponential function and we have,

if $y=\log_e x$

then $x=e^y$

Now $x=e^y$

$\therefore$ $\dfrac{dx}{dy}=e^y$

$\dfrac{dy}{dx}=\dfrac{1}{\dfrac{dx}{dy}}$

$=\dfrac{1}{e^y}=\dfrac{1}{x}$.

$\therefore$ if $y=\log_e x$, then $\dfrac{dy}{dx}=\dfrac{1}{x}$

3·6. Inverse Trigonometric Functions

The trigonometric functions form an important class of functions. Their inverses exist when certain restrictions are placed on the domains. It is not possible to discuss them in detail in a book on calculus but we introduce them to the student and obtain their derivatives.

1. $\sin^{-1} x$

Let $x = \sin y$ for $-\pi/2 \leqslant y \leqslant \pi/2$

Here x is given as a function of y. The domain is $[-\pi/2, \pi/2]$ and the range is $[-1, 1]$. The function is one-one and onto. Hence the inverse exists. The inverse relation is written as,

$$y = \sin^{-1} x, \quad -1 \leqslant x \leqslant 1, \quad -\pi/2 \leqslant y \leqslant \pi/2$$

Thus if $x = \sin y$ and $-\pi/2 \leqslant y \leqslant \pi/2$,

then $y = \sin^{-1} x$

Thus y represents the number lying between $-\pi/2$ and $\pi/2$ whose sine is x.

e.g. $\sin^{-1}(1) = \pi/2$, $\sin^{-1}(0) = 0$, $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = -\pi/4$.

The graph of $y = \sin^{-1} x$, $-1 \leqslant x \leqslant 1$; $-\pi/2 \leqslant y \leqslant \pi/2$ is shown in fig. 3·1.

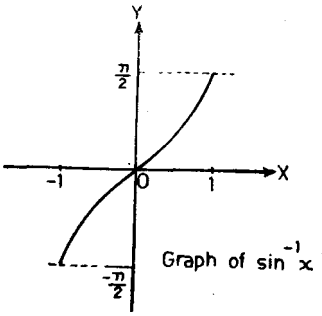

Fig. 3·1.

Now $x = \sin y$

$$\therefore \quad \frac{dx}{dy} = \cos y$$

$$= \sqrt{1 - x^2}$$

The sign being positive,

since $-\pi/2 \leqslant y \leqslant \pi/2$.

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \text{ if } \quad y=\sin^{-1}x, \text{ then } \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}.$$

2. $\cos^{-1}x$

Let $\quad x=\cos y \qquad 0\leqslant y\leqslant\pi$

x is given as a function of y. The domain of the function is $[0,\pi]$ and its range is $[-1, 1]$. The function is one-one and on to. Hence the inverse exists. The inverse relation is written as

$$y=\cos^{-1}x \qquad -1\leqslant x\leqslant 1\ ;\ 0\leqslant y\leqslant\pi$$

$\therefore$ if $\cos y=x$ and $0\leqslant y\leqslant\pi$,

then $\quad y=\cos^{-1}x.$

Thus $\cos^{-1}x$ is the number, lying between 0 and π whose cosine is x. *e.g.* $\cos^{-1}(0)=\frac{\pi}{2}$, $\cos^{-1}(1)=0$, $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$, $\cos^{-1}\left(-\frac{1}{2}\right)=2\frac{\pi}{3}$, $\cos^{-1}(-1)=\pi$ etc. The graph of $y=\cos^{-1}x$, is illustrated in fig. 3·2.

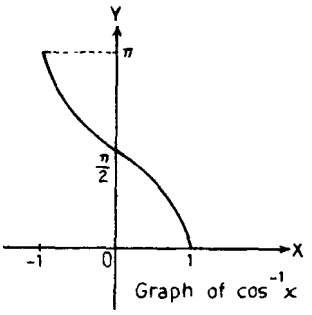

Fig. 3·2.

Now $\quad x=\cos y$

$$\therefore \quad \frac{dx}{dy}=-\sin y$$

$$=-\sqrt{1-x^2}$$

as $\quad \sin y>0$ since $\quad 0\leqslant y\leqslant\pi$

$$\therefore \quad \frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}$$

$\therefore$ if $\quad y=\cos^{-1}x$

then $\quad \frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}$

3. $\tan^{-1} x$

Let $x=\tan y$ for $-\pi/2<y<\pi/2$

Here again x is given as a function of y. The domain of the function is $(-\pi/2, \pi/2)$ and the range is $(-\infty, \infty)$. The function is one-one and onto. Hence the inverse exists. The inverse relation is written as,

$$y=\tan^{-1} x \quad \text{for} \quad -\infty<x<\infty$$

and $-\pi/2<y<\pi/2$

$\therefore$ if $\tan y=x$ and $-\pi/2\leqslant y<\pi/2$

then $y=\tan^{-1} x$

Thus $\tan^{-1} x$ is the number, lying between $-\pi/2$ and $\pi/2$ (and excluding these values) whose tangent is $x\in\mathbb{R}$. *e.g.* $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$, $\tan^{-1}(-1)=-\pi/4$, $\tan^{-1}(1)=\pi/4$. The graph of $y=\tan^{-1} x$, $x\in\mathbb{R}$ $-\pi/2<y<\pi/2$ is shown in fig. 3·3.

Now $x=\tan y.$

$$\therefore \quad \frac{dx}{dy}=\sec^2 x$$

$$=1+x^2$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{1+x^2}$$

$\therefore$ if $y=\tan^{-1} x$, then $\frac{dy}{dx}=\frac{1}{1+x^2}$

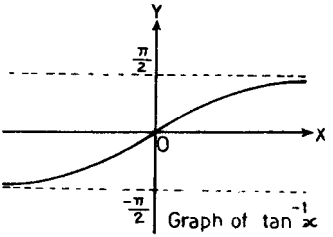

Fig. 3·3.

4. $\text{cosec}^{-1} x$

If $\text{cosec}\, y=x$ then as in the above cases, we can define the inverse of the cosecant function as,

$$y=\text{coses}^{-1} x \text{ for } |x| \geqslant 1 \text{ and } -\pi/2\leqslant y\leqslant\pi/2,\ y\neq 0$$

$\therefore$ if $\text{cosec}\, y=x$ and $-\pi/2\leqslant y\leqslant\pi/2$

then $y=\text{cosec}^{-1} x.$

Now $x = \text{cosec } y$

$$\therefore \quad \frac{dx}{dy} = -\text{cosec } y \,.\, \cot y$$

$$= -x\sqrt{x^2-1}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

$$\therefore \quad \text{if } y = \text{cosec}^{-1} x \text{ then } \frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

5. Sec ^{1}x

The inverse secant function is defined as,

$y = \sec^{-1}x$, if $| x | \geqslant 1$ and $0 \leqslant y \leqslant \pi$, $y \neq \pi/2$.

if $\sec y = x$,

$\therefore$ if $\sec y = x$ and $0 \leqslant y \leqslant \pi$, $y \neq \pi/2$.

then $y = \sec^{-1} x$

Now $x = \sec y$

$$\therefore \quad \frac{dx}{dy} = \sec y . \tan y$$

$$= x\sqrt{x^2-1}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

$$\therefore \quad \text{if } y = \sec^{-1} x \text{ then } \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

6. Cot^{-1}x

The inverse cotangent function is defined as

$y = \cot^{-1} x$ if $-\infty < x < \infty$ $-\pi/2 < y < \pi/2$ and $\cot y = x$

$\therefore$ if $\cot y = x$ and $-\pi/2 < y < \pi/2$

then $y = \cot^{-1} x$

Now $x = \cot y$

$$\therefore \quad \frac{dx}{dy} = -\text{cosec}^2 y$$

$$= -(1+x^2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{1+x^2}$$

$$\therefore \quad \text{if } y = \cot^{-1} x \text{ then } \frac{dy}{dx} = \frac{-1}{1+x^2}.$$

Illustrative Examples

To familiarise the student with the inverse trigonometric functions we first solve simple examples bearing on them and derive one or two important results.

Ex. 1. Write down the values of :

(*i*) $\sin^{-1}\frac{1}{2}$ (*ii*) $\sin^{-1}(-\frac{1}{2})$ (*iii*) $\sin^{-1}0$ (*iv*) $\cos^{-1}\frac{1}{2}$ (*v*) $\cos^{-1}(-\frac{1}{2})$ (*vi*) $\cos^{-1}(-1)$ (*vii*) $\tan^{-1}\sqrt{3}$ (*viii*) $\tan^{-1}(-1)$ (*iv*) $\tan^{-1}0$.

Sol. (*i*) $\sin^{-1}\frac{1}{2} = \pi/6$ since $\pi/6 = \frac{1}{2}$ and $\pi/2 \leqslant \pi/6 \leqslant \pi/2$.

(*ii*) $\sin^{-1}(-\frac{1}{2}) = -\pi/6$ since $\sin(-\pi/6) = -\frac{1}{2}$ and $-\pi/2 \leqslant -\pi/6 \leqslant \pi/2$.

(*iii*) $\sin^{-1}0 = 0$ since $\sin 0 = 0$ and $\pi/2 \leqslant 0 \leqslant \pi/2$.

(*iv*) $\cos^{-1}\frac{1}{2} = \pi/3$ since $\cos \pi/3 = \frac{1}{2}$ and $0 \leqslant \pi/3 \leqslant \pi$.

(*v*) $\cos^{-1}(-\frac{1}{2}) = 2\pi/3$ since $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $0 \leqslant 2\pi/3 \leqslant \pi$.

(*vi*) $\cos^{-1}(-1) = \pi$ since $\cos \pi = -1$ and $0 \leqslant \pi \leqslant \pi$.

(*vii*) $\tan^{-1}\sqrt{3} = \pi/3$ since $\tan \pi/3 = \sqrt{3}$ and $-\pi/_2 < \pi/3 < \pi/_3$.

(*viii*) $\tan^{-1}(-1) = -\pi/4$ since $\tan(-\pi/4) = -1$ and $-\pi/2 < \pi/_4 < \pi/_2$.

(*ix*) $\tan^{-1}0 = 0$ since $\tan 0 = 0$ and $-\pi/2 \leqslant 0 \leqslant \pi/2$.

Ex. 2. Prove that $\sin^{-1}x + \cos^{-1}x = \pi/2$.

Sol. Let $\sin^{-1}x = \alpha$ $\therefore$ $\sin\alpha = x$ and $-\pi/2 \leqslant \alpha \leqslant \pi/2$ ...(1).

$\cos^{-1}x = \beta$ $\therefore$ $\cos\beta = x$ and $0 \leqslant \beta \leqslant \pi$ ...(2)

$\therefore$ $\cos\alpha = \sqrt{1-x^2}$ positive $\because$ $-\pi/2 \leqslant x \leqslant \pi/2$.

$\therefore$ $\sin\beta = \sqrt{1-x^2}$ positive $\because$ $0 \leqslant \beta \leqslant \pi$.

$\therefore$ $\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$

$= x \cdot x + \sqrt{1-x^2}\ \sqrt{1-x^2}$

$= x^2 + 1 - x^2$

$= 1$

$\therefore$ $\sin(\alpha+\beta) = 1$ and from (1) and (2),

$$-\pi/_2 \leqslant \alpha+\beta \leqslant \frac{3\pi}{2}$$

$\therefore$ $\alpha+\beta = \pi/2$, since $\pi/2$ is the only angle between $-\pi/2$ and $3\pi/2$ such that its sine is 1,

$\therefore$ $\sin^{-1}x + \cos^{-1}x = \pi/2$.

Ex. 3. It can also be proved that, (*a*) if $x > 0$, $y > 0$ and $xy < 1$, $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(*b*) if $x > 0$, $y > 0$ then $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Use these results to prove the following :

(*i*) $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}$

(*ii*) $\tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \pi/4$

(*iii*) $\tan^{-1}\frac{7}{4} \tan^{-1}\frac{1}{6} = \tan^{-1}\frac{38}{31}$

Sol.

(*i*) $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\dfrac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\cdot\frac{1}{13}}$ using(*a*)

$= \tan^{-1}\dfrac{20/91}{90/91}$

$= \tan^{-1}\dfrac{2}{9}$

(*ii*) $\tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \tan^{-1}\dfrac{\frac{3}{5}+\frac{1}{4}}{1-\frac{3}{5}\cdot\frac{1}{4}}$ using (*a*)

$= \tan^{-1}\dfrac{17/20}{17/20}$

$= \tan^{-1} 1 = \dfrac{\pi}{4}$

(*iii*) $\tan^{-1}\frac{7}{4} - \tan^{-1}\frac{1}{6} = \tan^{-1}\dfrac{\frac{7}{4}-\frac{1}{6}}{1+\frac{7}{4}\cdot\frac{1}{6}}$ using (*b*)

$= \tan^{-1}\dfrac{38}{31}$

Ex. 3. If $0 < x < 1$, show that

$$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Sol. By (*a*) above since $x \cdot x < 1$

$$\tan^{-1} + \tan^{-1} x = \tan^{-1}\frac{x+x}{1-x\cdot x}$$

i.e., $2\tan^{-1} x = \tan^{-1}\dfrac{2x}{1-x^2}$

Ex. 4. If $\sin^{-1} x = \phi$, show that
$\sin^{-1} (3x-4x^3) = 3\phi$ if $-\pi/2 \leqslant 3\phi \leqslant \pi/2$

Sol. $\sin^{-1} x = \phi$

$\therefore \sin \phi = x$

$\therefore \sin 3\phi = 3 \sin \phi - 4 \sin^3 \phi$

$= 3x - 4x^3$ and $-\pi/2 \leqslant 3\phi \leqslant \pi/2$

$\therefore 3\phi = \sin^{-1} (3x-4x^3)$

Ex. 5. Find $\frac{dy}{dx}$ in the following cases,

(*i*) $y = \sin^{-1} 2x$ (*ii*) $y = \cos^{-1} a x$ (*iii*) $y = \tan^{-1} (\frac{1}{2} \cdot x)$

Sol.

(*i*) $y = \sin^{-1} 2x$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$$

$$= \frac{2}{\sqrt{1-4x^2}}.$$

(*ii*) $y = \cos^{-1} ax$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-(ax)^2}} \cdot a$$

$$= \frac{-a}{\sqrt{1-a^2x^2}}$$

(*iii*) $y = \tan^{-1} (\frac{1}{2} \cdot x)$

$$\therefore \frac{dy}{dx} = \frac{1}{1+(\frac{1}{2}x)^2} \cdot \frac{1}{2}$$

$$= \frac{2}{4+x^2}$$

Ex. 6. Find the derivatives of

(*i*) $\sin^{-1} \sqrt{1-x^2}$ (*ii*) $\sin^{-1} (3x-4x^3)$ (*iii*) $\tan^{-1} \frac{2x}{1-x^2}$

Sol. (*i*) $y = \sin^{-1} \sqrt{1-x^2}$

Let $\phi = \cos^{-1} x$ $\quad \therefore \frac{d\phi}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$\therefore \cos \phi = x$

$\therefore \sin \phi = \sqrt{1-x^2}$

$$\therefore \quad y = \sin^{-1}(\sin\phi)$$
$$= \phi \qquad \text{if } -\pi/2 \leqslant \phi \leqslant \pi/2$$
$$\therefore \quad \frac{dy}{dx} = \frac{d\phi}{dx}$$
$$= \frac{-1}{\sqrt{1-x^2}}$$

(*ii*) $y = \sin^{-1}(3x-4x^3)$

Let $\phi = \sin^{-1}x \qquad \therefore \quad \frac{d\phi}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \quad x = \sin\phi$$
$$\therefore \quad 3x-4x^3 = 3\sin\phi - 4\sin^3\phi$$
$$= \sin 3\phi$$
$$\therefore \quad y = \sin^{-1}(\sin 3\phi)$$
$$= 3\phi \qquad \text{if } -\pi/2 \leqslant 3\phi \leqslant \pi/2$$
$$\therefore \quad \frac{dy}{dx} = 3 \cdot \frac{d\phi}{dx}$$
$$= \frac{3}{\sqrt{1-x^2}}$$

(*iii*) $y = \tan^{-1}\frac{2x}{1-x^2}$

Let $\phi = \tan^{-1}x \qquad \therefore \quad x = \tan\phi$

$$\therefore \quad \frac{d\phi}{dx} = \frac{1}{1+x^2}$$

Now $\frac{2x}{1-x^2} = \frac{2\tan\phi}{1-\tan^2\phi}$

$$= \tan 2\phi$$
$$\therefore \quad y = \tan^{-1}(\tan 2\phi)$$
$$= 2\phi \qquad \text{if } -\pi/2 < 2\phi < \pi/2$$
$$\therefore \quad \frac{dy}{dx} = 2 \cdot \frac{d\phi}{dx}$$
$$= \frac{2}{1+x^2} \qquad \therefore$$

Note. In the above solution we have restricted the values of ϕ suitably to simplify calculations. You can verify that the results

obtained are generally true. For example, in (*i*) we may proceed as follows.

$$y=\sin^{-1}\sqrt{1-x^2}$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\sqrt{1-(1-x^2)}}\times\frac{1}{2}\times\frac{-2x}{\sqrt{1-x^2}}$$

$$=\frac{1}{\sqrt{x^2}}\times\frac{-x}{\sqrt{1-x^2}}$$

$$=\frac{-1}{\sqrt{1-x^2}},$$

as obtained above. We shall hence forward omit the restrictions on ϕ.

Ex. 7. Find the derivatives of,

(*i*) $\tan^{-1}\left(\dfrac{2-3x}{1+6x}\right)$ (*ii*) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(*iii*) $\sec^{-1}\left(\dfrac{1}{8x^2-1}\right)$

Sol. (*i*) $y=\tan^{-1}\dfrac{2-3x}{1+6x}$

$$=\tan^{-1}2-\tan^{-1}3x$$

$$\therefore \quad \frac{dy}{dx}=\frac{-3}{1+9x^2}$$

(*ii*) $y=\cos^{-1}\dfrac{1-x^2}{1+x^2}$

Put $\phi=\tan^{-1}x$ *i.e.* $\tan\phi=x$

$$\therefore \quad \frac{1-x^2}{1+x^2}=\frac{1-\tan^2\phi}{1+\tan^2\phi}$$

$$=\cos 2\phi$$

$$\therefore \quad y=\cos^{-1}(\cos 2\phi)$$

$$=2\phi.$$

$$\therefore \quad \frac{dy}{dx}=2\frac{d\phi}{dx}$$

$$=\frac{2}{1+x^2} \qquad \because \quad \phi=\tan^{-1}x$$

(*iii*) $y=\sec^{-1}\left(\dfrac{1}{8x^2-1}\right)$

Put $\phi=\cos^{-1}2x$

$$\therefore \quad \cos\phi=2x$$

and $$\frac{d\phi}{dx}=\frac{-1}{\sqrt{1-4x^2}}$$

$\therefore$ $$8x^2-1=2\cos^2\phi-1$$
$$=\cos 2\phi$$

$\therefore$ $$\frac{1}{8x^2-1}=\sec 2\phi$$

$\therefore$ $$y=\sec^{-1}(\sec 2\phi)$$
$$=2\phi$$

$\therefore$ $$\frac{dy}{dx}=2.\frac{d\phi}{dx}$$
$$=\frac{-2}{\sqrt{1-4x^2}}$$

Ex. 8. If $y=\sin^{-1}\left(\frac{a\sin x+b\cos x}{\sqrt{a^2+b^2}}\right)$ find $\frac{dy}{dx}$.

Sol. Let α be such that,

$$\tan\alpha=\frac{b}{a}$$

$\therefore$ $$\cos\alpha=\frac{a}{\sqrt{a^2+b^2}},\ \sin\alpha=\frac{b}{\sqrt{a^2+b^2}}$$

$\therefore$ $$y=\sin^{-1}[\sin x\cos\alpha+\cos x\sin\alpha]$$
$$=\sin^{-1}[\sin(x+\alpha)]$$

$\therefore$ $$y=x+\alpha$$

$\therefore$ $$\frac{dy}{dx}=1.$$

EXERCISE

Inverse Functions

1. If $x=\sqrt{\sin y}$ for $0\leqslant y\leqslant \pi/2$, find $\frac{dy}{dx}$ as a function of x.

2. If x is expressed as a function of y by the relation,

$$x=\frac{1}{\sqrt{1+y^2}},\ x\geqslant 0,\ y\geqslant 0$$

Show that the inverse exists. Find $\frac{dx}{dy}$ and hence or otherwise show that,

$$y\frac{dy}{dx}+x^3=0$$

3. Differentiate w.r.t. x.

(*i*) $\sin^{-1}(x+1)$ (*ii*) $\cos^{-1}(2-x)$ (*iii*) $\tan^{-1}x^2$

(*iv*) $\sec^{-1}2x$ (*v*) $\operatorname{cosec}^{-1}(2x+1)$ (*vi*) $\cot^{-1}\left(\frac{1}{x}\right)$

(*vii*) $\tan^{-1}\sqrt{1+x^2}$ (*viii*) $\sin^{-1}\sqrt{1-2x^2}$

4. Find the derivatives of

(i) $\cos^{-1}\sqrt{1-x^2}$ (ii) $\tan^{-1}\sqrt{x^2-1}$ (iii) $\sec^{-1}\sqrt{1+x^2}$

(iv) $\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$ (v) $\sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$ (vi) $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

(vii) $\sin^{-1}(2x\sqrt{1-x^2})$ $(viii)$ $\cos^{-1}(2x^2-1)$ (ix) $\tan^{-1}\dfrac{2x}{1-x^2}$

(x) $\cos^{-1}(4x^3-3x)$ (xi) $\sin^{-1}\dfrac{1}{\sqrt{2}}(x+\sqrt{1-x^2})$

(xii) $\cos^{-1}\frac{1}{2}(\sqrt{3}x+\sqrt{1-x^2})$

$(xiii)$ $\sin^{-1}(x\sin\alpha+\cos\alpha\sqrt{1-x^2})$

5. Differentiate w.r.t. x.

(i) $\sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$ (ii) $\tan^{-1}\left(\dfrac{x+3}{1-3x}\right)$ (iii) $\tan^{-1}\left(\dfrac{x^2-a}{1+ax^2}\right)$

(iv) $\tan^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ (v) $\cot^{-1}\left(\dfrac{a+x}{1-ax}\right)$ (vi) $\cot^{-1}\sqrt{\dfrac{1-x}{1+x}}$

(vii) $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$ $(viii)$ $\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$ (ix) $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$

(x) $\cot^{-1}\left(\dfrac{x-1}{x+1}\right)$ (xi) $\sin^{-1}(\sin x)^{1/2}$ (xii) $\cot^{-1}\dfrac{e^x-e^{-x}}{2}$

ANSWERS

1. $\dfrac{2x}{\sqrt{1-x^2}}$

3. (i) $\dfrac{1}{\sqrt{-x^2-2x}}$ (ii) $\dfrac{-1}{(2-x)\sqrt{x^2-4x+3}}$ (iii) $\dfrac{2x}{1+x^4}$

(iv) $\dfrac{1}{x\sqrt{4x^2-1}}$ (v) $\dfrac{-2}{(2x+1)\sqrt{4x^2+4x}}$ (vi) $\dfrac{1}{1+x^2}$

(vii) $\dfrac{x}{(2+x^2)\sqrt{1+x^2}}$ $(viii)$ $\dfrac{-\sqrt{2}}{\sqrt{1-2x^2}}$

4. (i) $\dfrac{1}{\sqrt{1-x^2}}$ (ii) $\dfrac{1}{x\sqrt{x^2-1}}$ (iii) $\dfrac{1}{1+x^2}$

(iv) $\dfrac{1}{1+x^2}$ (v) $\dfrac{1}{1+x^2}$ (vi) $\dfrac{1}{\sqrt{1-x^2}}$

(vii) $\dfrac{2}{\sqrt{1-x^2}}$ $(viii)$ $\dfrac{-2}{\sqrt{1-x^2}}$ (ix) $\dfrac{2}{1+x^2}$

(x) $\dfrac{-3}{\sqrt{1-x^2}}$ (xi) $\dfrac{1}{\sqrt{1-x^2}}$ (xii) $\dfrac{-1}{\sqrt{1-x^2}}$

$(xiii)$ $\dfrac{1}{\sqrt{1-x^2}}$

5. (i) $\dfrac{2}{1+x^2}$ (ii) $\dfrac{1}{1+x^2}$ (iii) $\dfrac{2x}{1+x^4}$

(iv) $\dfrac{-2x}{1+x^4}$ (v) $\dfrac{-1}{1+x^2}$ (vi) $\dfrac{-1}{2\sqrt{1-x^2}}$

(*vii*) $\frac{1}{2}$ (*viii*) $-\frac{1}{2}$ (*ix*) $\frac{-2}{\sqrt{1-x^2}}$

(*x*) $\frac{-1}{1+x^2}$ (*xi*) $\frac{\cos x}{2\sqrt{\sin x\,(1-\sin x)}}$

(*xii*) $\frac{2}{e^x+e^{-x}}$

3·7. Logarithmic Differentiation.

Whenever a function is the product or quotient of a number of factors or is of the form $[f(x)]^{\varphi(x)}$ it is usually convenient to differentiate the logarithm of the function to find its derivative. This method of obtaining derivatives of functions is known as *logarithmic differentiation*. The following examples illustrate the method.

Ex. 1. $y=a^x$

Sol. $y=a^x$

$\therefore$ $\log y=x \log a$

Differentiating both sides w.r.t. x we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\log a$$

$$\therefore \quad \frac{dy}{dx}=y \log a$$

$$\therefore \quad \frac{dy}{dx}= a^x \log a$$

Ex. 2. $y = \dfrac{(x^3+1)\sqrt{x^2+x+1}}{(x^2+3x+5)(x^4+1)}$

Sol. $\log y = \log (x^3+1)+\frac{1}{2}\log (x^2+x+1)$

$$-\log (x^2+3x+5)-\log (x^4+1)$$

Differentiating both the sides *w.r.t* x we get,

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{3x^2}{x^3+1}+\frac{2x+1}{2(x^2+x+1)}-\frac{2x+3}{x^2+3x+5}-\frac{4x^3}{x^4+1}$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{3x^2}{x^3+1}+\frac{2x+1}{2(x^2+x+1)}-\frac{2x+3}{x^2+3x+5}-\frac{4x^3}{x^4+1}\right]$$

$$=\frac{(x^3+1)\sqrt{x^2+x+1}}{(x^2+3x+5)(x^4+1)}\left[\frac{3x^2}{x^3+1}+\frac{2x+1}{2(x^2+x+1)}-\frac{2x+3}{x^2+3x+5}-\frac{4x^3}{x^4+1}\right]$$

Ex. 3. $y = a^{x\sin x}$

Sol. $y = a^{x\sin x}$

$\therefore$ $\log y = \log a^{x\sin x}$

$= x \sin x \log a$

Differentiating both the sides *w.r.t.* x, we get,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \log a + x \cos x \log a$$

$$\therefore \quad \frac{dy}{dx} = y\,[\sin x \log a + x \cos x \log a]$$

$$= a^{x \sin x} \log a\,[\sin x + x \cos x\,]$$

Ex. 4. $y = (x \sin x + \cos x)^{(\sin x + \cos x)}$

Sol. $\log y = \log (x \sin x + \cos x)^{(\sin x + \cos x)}$

$$= (\sin x + \cos x) \log (x \sin x + \cos x)$$

Differentiating both sides *w.r.t.* x we get,

$$\frac{1}{y} \cdot \frac{dy}{dx} = (\cos x - \sin x) \log (x \sin x + \cos x) + (\sin x + \cos x) \times \frac{(x \cos x + \sin x - \sin x)}{(x \sin x + \cos x)}$$

$$= (\cos x - \sin x) \log (x \sin x + \cos x) + \frac{x \cos x\,(\sin x + \cos x)}{(x \sin x + \cos x)}$$

$$\therefore \quad \frac{dy}{dx} = y\,[(\cos x - \sin x) \log (x \sin x + \cos x) + \frac{x \cos x\,(\sin x + \cos x)}{(x \cos x + \sin x)}\,]$$

$$\therefore \quad \frac{dy}{dx} = (x \sin x + \cos x)^{(\sin x + \cos x)}[(\cos x - \sin x) \log (x \sin x + \cos x) \frac{x \cos x\,(\sin x + \cos x)}{(x \sin x + \cos x)}\,]$$

Ex. 5. $y = x^x$

Sol. $\therefore \quad \log y = \log x^x$

$$= x \log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log x + x \cdot \frac{1}{x}$$

$$= \log x + 1$$

$$\therefore \quad \frac{dy}{dx} = y\,(\log x + 1)$$

$$= x^x\,(\log x + 1)$$

Ex. 6. $y = e^{x^x}$

Sol. $y = e^{x^x}$

$$\therefore \log y = x^x \log e$$
$$= x^x$$
$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = x^x (\log x + 1)$$
$$\therefore \frac{dy}{dx} = y \cdot x^x (\log x + 1)$$
$$= e^{x^x} \cdot x^x (\log x + 1)$$

Ex. 7. $y = (\sin x)^{\cos x}$

Sol. $\therefore \log y = \log (\sin x)^{\cos x}$
$= \cos x \log \sin x$

Differentiating both sides w.r.t. x, we get,

$$\frac{1}{y} \frac{dy}{dx} = -\sin x \log \sin x + \frac{\cos x}{\sin x} \cdot \cos x$$
$$= \frac{\cos^2 x - \sin^2 x \log \sin x}{\sin x}$$
$$\therefore \frac{dy}{dx} = y \cdot \frac{\cos^2 x - \sin^2 x \log \sin x}{\sin x}$$
$$= (\sin x)^{\cos x} \frac{\cos^2 x - \sin^2 x \log \sin x}{\sin x}$$

EXERCISE

(Logarithmi Differentiation.) Differentiate w.r.t. x

1. (*i*) $\frac{x(1+x^2)}{\sqrt{1-x^2}}$ (*ii*) $\frac{(x+1)^2}{(x+2)^3 (x+3)^4}$ (*iii*) $\sqrt{\frac{x(x+1)}{x-1}}$

(*iv*) $\frac{\sqrt[3]{x(1+x^2)}}{(x^2-1)}$

2. (*i*) $\sin x^{\tan x}$ (*ii*) a^{x^x} (*iii*) $x^{x \sin x}$

(*iv*) $(\log x)^{\sin^2 x}$ (*v*) $(\sin x)^x + x^{\sin x}$

(*vi*) $x^{\sqrt{x}}$ (*vii*) $x^{\log (\log x)}$

(*viii*) x^{a^x} (*ix*) $(\log x)^{\log x}$ (*x*) $e^{3/x}$

(*xi*) $(\cos x)^{\sin x}$ (*xii*) $(\tan x)^{\sec x}$ (*xiii*) x^{x^x}

3. (*i*) $(1+x^2)^{x \sin x}$ (*ii*) $x^{\sin^{-1} x} \cdot (1+x^2)^x$

(*iii*) $(2x+3)^{x^2+1}$ (*iv*) $(x+1)^{1/x}$ (*v*) $(x^2+x+1)^{x-1}$,

(*vi*) $(x^2+3)^{\sin x}$ (*vii*) $\left(\frac{x}{x+1}\right)^x$ (*viii*) $(1-x^2)^{x+1}$

(*ix*) $(\sin x + \cos x)^{(\tan x + 1)}$

ANSWERS

1. (*i*) $\frac{1+3x^2-2x^4}{x(1+x^2)(1-x^2)}$ (*ii*) $\frac{-(5x^2+14x+5)}{(x+1)(x+2)(x+3)}$

(*iii*) $\frac{x^2-2x-1}{2x(x^2-1)}$ (*iv*) $\frac{x^4-4x^2-1}{3x(x^4-1)}$

2. (*i*) $\sin x^{\tan x}\,[\sec^2 x \log \sin x+1]$

(*ii*) $a^{x^x} \log a\,(\log x+1)x^x$

(*iii*) $x^{x \sin x}\,[\sin x\,(\log+1)+x \cos x.\ \log x]$

(*iv*) $(\log x)^{\sin^2 x}\left(\frac{\sin^2 x}{x}+\sin 2x \log x\right)$

(*v*) $(\sin x)^x\ [\log \sin x+x \cot x]+x^{\sin x}[\cos x \log x+\frac{\sin x}{x}\Big)$

(*vi*) $x^{x^{1/2}-\frac{1}{2}}\left(\frac{1}{2}\log x+1\right)$

(*vii*) $x^{\log(\log x)-1}[1+\log \log x]$

(*viii*) $x^{a^x}.a^x\left[\log a.\log x+\frac{1}{x}\right]$

(*ix*) $(\log x)^{\log x}\,\frac{2 \log x}{x}$

(*x*) $-3x^{-2}\,e^{3/x}$ (*xi*) $(\cos x)^{\sin x}[\cos x \log \cos x-\tan x \sin x]$

(*xii*) $(\tan x)^{\sec x} \sec x\ [\operatorname{cosec} x.\sec x+\tan x \log\tan x]$

(*xiii*) $x^{x^x}.[x^{x-1}+x^x \log x\,(\log x+1)]$

3. (*i*) $(1+x^2)^{x \sin x}\ [\sin x \log(1+x^2)+x \cos x \log(1+x^2)$

$$+\frac{2x^2 \sin x}{1+x^2}\Big]$$

(*ii*) $x^{\sin^{-1}x}\ (1+x^2)^x\left[\frac{\log x}{\sqrt{1-x^2}}+\frac{\sin^{-1}x}{x}+\log(1+x^2)+\frac{2x^2}{1+x^2}\right]$

(*iii*) $(2x^2+3)^{(x^2+1)}\left[2x \log(2x+3)+\frac{4x(x^2+1)}{2x^2+3}\right]$

(*iv*) $(x+1)^{1/x}\ \left[\frac{1}{x(x+1)}-\frac{\log(x+1)}{x^2}\right]$

(*v*) $(x^2+x+1)^{(x-1)}\ \left[\log(x^2+x+1)+\frac{(x-1)(2x+1)}{x^2+x+1}\right]$

(*vi*) $(x^2+3)^{\sin x}\left[\cos x \log(x^2+3)+\frac{2x \sin x}{x^2+3}\right]$

(*vii*) $\left(\frac{x}{x+1}\right)^x\left[\log\frac{x}{x+1}+\frac{1}{x+1}\right]$

(*viii*) $(1-x^2)^{x+1}\left[\log(1-x^2)-\frac{2x(x+1)}{1-x^2}\right]$

(*ix*) $(\sin x+\cos x)^{(1+\tan x)}[\sec^2 x \log(\sin x+\cos x)$
$+\sec x(\cos x-\sin x)]$

38. Implicit Functions

Sometimes y is defined as a function of x implicitly by a relation of the type $\phi(x, y)=0$.

In such a case we differentiate the relation w.r.t. x and obtain $\frac{dy}{dx}$ by solving the resulting equation. The following examples will make the procedure clear.

Ex. 1. $x^2+y^2+2x+4y+5=0$

Sol. Differentiating w.r.t. x, we obtain,

$$2x+2y\frac{dy}{dx}+2+4\frac{dy}{dx}=0$$

$$\therefore \quad x+y\frac{dy}{dx}+1+2\frac{dy}{dx}=0$$

$$\therefore \quad (y+2)\frac{dy}{dx}=-(x+1)$$

$$\therefore \quad \frac{dy}{dx}=\frac{-(x+1)}{(y+2)} \qquad y\neq -2$$

Ex. 2. $y=\cos(x+y)$

Sol. Differentiating w.r.t. x we get,

$$\frac{dy}{dx}=-\sin(x+y)\left[1+\frac{dy}{dx}\right]$$

$$=-\sin(x+y)-\sin(x+y)\frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx}[1+\sin(x+y)]=-\sin(x+y)$$

$$\therefore \quad \frac{dy}{dx}=\frac{-\sin(x+y)}{1+\sin(x+y)} \qquad 1+\sin(x+y)\neq 0$$

Ex. 3. $x+y=\sqrt{xy}$

Sol. Squaring we get,

$$(x+y)^2=xy$$

$$\therefore \quad x^2+2xy+y^2=xy$$

$$\therefore \text{ i.e.} \quad x^2+xy+y^2=0$$

Differentiating w.r.t. x we get

$$2x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0$$

$$\therefore \quad (x+2y)\frac{dy}{dx} = -(2x+y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$$

Ex. 4. $x^y = e^{x-y}$.

Sol. Taking logarithms of both sides, we get

$$\log x^y = \log e^{x-y}$$

$$y \log x = (x-y) \qquad \ldots(A)$$

Differentiating both sides w.r t. x we get

$$\frac{dy}{dx} \log x + \frac{y}{x} = 1 - \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx}(\log x+1) = 1 - \frac{y}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x-y}{x(\log x+1)}$$

Now from (A) above, we obtain,

$$x = y(\log x+1) \qquad \ldots(B)$$

$$\therefore \quad \frac{dy}{dx} = \frac{y \log x}{y(\log x+1)(\log x+1)} \text{ using } (A) \text{ and } (B)$$

$$= \frac{\log x}{(\log x+1)^2}$$

EXERCISE *(Implicit Functions)*

1. Find $\frac{dy}{dx}$ if :

(*i*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(*ii*) $x^2+y^2+4x-6y+2=0$

(*iii*) $2x^2+5xy+3y^2+2x+4y+1=0$

(*iv*) $x^{1/2}+y^{1/2} = a^{1/2}$ (*v*) $x^3+y^3-3axy=0$

(*vi*) $x^{2/3}+y^{2/3} = a^{2/3}$ (*vii*) $y = \cos(x+y)$

(*viii*) $\cos(xy) = x$ (*ix*) $x = y \tan^{-1} y$

(*x*) $\sin(xy) = x \cos y$.

2 If $x^m.y^n = (x+y)^{m+n}$, show that

$$\frac{dy}{dx} = \frac{y}{x}$$

3. If $y = x.e^{xy}$, show that $\frac{dy}{dx} = \frac{y}{x}\frac{1+xy}{1-xy}$

4. If $\cos xy = \sin(x+y)$, show that,

$$\frac{dy}{dx} = \frac{y \sin xy - \cos(x+y)}{x \sin xy + \cos(x+y)}$$

5. If $y=xe^y$ show that $\frac{dy}{dx}=\frac{y/x}{1-\log\left(\frac{y}{x}\right)}$

6. If $e^{x+y}=e^{xy}+k$, show that, $\frac{dy}{dx}=-\frac{1-y}{1-x}$

7. If $y=x\cos y$, show that $\frac{dy}{dx}=\frac{\cos^2 y}{\cos y+y\sin y}$

ANSWERS

(*i*) $\frac{-b^2x}{a^2y}$ (*ii*) $-\frac{x+2}{y-3}$ (*iii*) $-\frac{4x+5y+2}{5x+6y+4}$

(*iv*) $\sqrt{\frac{y}{x}}$ (*v*) $\frac{ay-x^2}{y^2-ax}$ (*vi*) $\sqrt[3]{y/x}$

(*vii*) $-\frac{\sin(x+y)}{1+\sin(x+y)}$ (*viii*) $\frac{1+y\sin(xy)}{x\sin(xy)}$

(*ix*) $1+\frac{1}{y^2}$ (*x*) $\frac{\cos y-y\cos(xy)}{x[\cos(xy)+\sin y]}$

3·9. Parametric Equations

Many times the functional relation between x and y is expressed in terms of a third variable called a parameter *e.g.* the co-ordinates (x, y) of a point on the parabola $y^2=4ax$ can be expressed as

$$\left.\begin{aligned}x&=at^2\\y&=2at\end{aligned}\right\}\ t\in\mathbb{R}\ \text{ or}$$

the equation of the circle $x^2+y^2=a^2$ can be expressed as

$$\left.\begin{aligned}x&=a\cos\theta\\y&=a\sin\theta\end{aligned}\right\}\ 0\leqslant\theta<2\pi$$

such equations are called parametric equations t, θ etc being the parameters.

To find $\frac{dy}{dx}$ in such cases, we note that as the change δt in t tends to 0 so do the changes δx and δy in x and y respectively using the fact

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta t}\div\frac{\delta x}{\delta t}$$

we get

$$\lim_{\delta t\to 0}\left(\frac{\delta y}{\delta x}\right)=\lim_{\delta t\to 0}\left(\frac{\delta y}{\delta t}\div\frac{\delta x}{\delta t}\right)$$

$$\therefore\quad \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)=\frac{\lim\limits_{\delta t\to 0}\left(\frac{\delta y}{\delta t}\right)}{\lim\limits_{\delta\to 0}\left(\frac{\delta x}{\delta t}\right)}$$

$$\therefore\quad \frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}$$

Illustrative Examples

Ex. 1. Find $\frac{dy}{dx}$ in each of the following cases

(*i*) $x=at^2 \quad y=2at$, (*ii*) $x=a\cos\theta, y=a\sin\theta$

(*iii*) $x=\cos(\log t)\ y=\log(\cos t)$

Sol. (*i*) $x=at^2, \frac{dx}{dt}=2at$

$$y=2at,\ \frac{dy}{dt}=2a$$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2a}{2at}=\frac{1}{t}$$

(*ii*) $x=a\cos\theta \quad \therefore \quad \frac{dx}{d\theta}=-a\sin\theta$

$$y=a\sin\theta \quad \therefore \quad \frac{dy}{d\theta}=a\cos\theta$$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a\cos\theta}{-a\sin\theta}=-\cot\theta$$

(*iii*) $x=\cos(\log t) \quad \therefore \quad \frac{dx}{dt}=-\sin(\log t)\times\frac{1}{t}$

$$y=\log(\cos t) \quad \therefore \quad \frac{dy}{dx}=\frac{1}{\cos t}(-\sin t)$$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{-\sin t}{\cos t}}{\frac{-\sin(\log t)}{t}}$$

$$=\frac{t\sin t}{\cos t\sin(\log t)}$$

Ex. 2. Differentiate $x\sin x$ w.r.t. $\tan x$.

Sol. Let $u=x\sin x$ and $v=\tan x$

$\therefore$ u and v are expressed in terms of x as a parameter.

$$\therefore \quad \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}} \qquad \ldots(1)$$

Now $\frac{du}{dx} = x \cos x + \sin x$

$\frac{dv}{dx} = \sec^2 x$

$\therefore$ by (1) ; $\frac{du}{dv} = -\frac{x \cos x + \sin x}{\sec^2 x}$

$= (x \cos x + \sin x) \cos^2 x$

EXERCISE

Find $\frac{dy}{dx}$ in the following :

1. $x = a \cos \theta$, $y = b \sin \theta$
2. $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$
3. $x = \frac{a}{m^2}$ $y = \frac{2a}{m}$
4. $x = e^{\cos 2t}$, $y = e^{\sin 2t}$
5. $x = 3 \sin t - 2 \sin^3 t$, $y = 3 \cos t - 2 \cos^3 t$
6. Differentiate $\tan^{-1} x$ w.r.t. $\frac{1}{1+x^2}$
7. Differentiate $x \sin 2x$ w.r.t. $\tan^{-1} x$
8. Find the derivative of $\tan^{-1} x$ w.r.t. $\tan x$
9. Differentiate $e^{\cos x}$ w.r.t. $\sin^2 x$
10. Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

ANSWERS

1. $\frac{-b \cot \theta}{a}$ 2. $\cot \frac{\theta}{2}$ 3. m 4. $\frac{-y}{x} \frac{\log x}{\log y}$
5. $\cot t$ 6. $\frac{-(1+x^2)}{2x}$ 7. $(1+x^2)(2x \cos 2x + \sin 2x)$
8. $\frac{\cos^2 x}{1+x^2}$ 9. $-\frac{1}{2} e^x \sec x$ 10. 1

3·10. Derivatives of higher order

Let $f'(x)$ be the derivative of a function f where x belongs to the domain I of f. We can look upon $f'(x)$ as another function of x say $g(x)$. If

$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$, exists for all $x \in$ I then, this limit is called the second derivative of f at x and is denoted by $f''(x)$. If $y = f(x)$ then $f'(x) = \frac{dy}{dx}$ and $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ which is denoted as $\frac{d^2y}{dx^2}$. In practice we obtain the second derivative by the same methods that

we employ for getting a derivative of a function. Thus if $y = \sin x$, $\frac{dy}{dx} = \cos x$ and $\frac{d^2y}{dx^2} = -\sin x$.

The process can be extended to get derivatives of higher order also, if they exist.

Illustrative Examples

Ex. 1. If $y = e^{ax} \sin (bx+c)$ find $\frac{d^2y}{dx^2}$

Sol. $$\frac{dy}{dx} = e^{ax}\, b \cos (bx+c) + ae^{ax} \sin (bx+c)$$

$$\frac{d^2y}{dx^2} = e^{ax} \{-\sin (bx+c)\}\, b^2 + ae^{ax}\, b \cos (bx+c)$$

$$+ ae^{ax} \cos (bx+c)\, b + a^2e^{ax} \sin (bx+c)$$

$$= e^{ax} \sin (bx+c)\,(a^2-b^2) + 2b\, ae^{ax} \cos (bx+c)$$

Ex. 2. If $y^2 = ax^2+2bx+c$, where a,b,c are constants, show that $y^3 \frac{d^2y}{dx^2}$ is constant.

Sol. Differentiating w.r.t. x both sides, we get

$$2y \frac{dy}{dx} = 2ax+2b$$

$$\therefore \quad \frac{dy}{dx} = \frac{ax+b}{y}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{y - a - (ax+b)\frac{dy}{dx}}{y^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{ay - (ax+b)\left(\frac{ax+b}{y}\right)}{y^2}$$

$$= \frac{ay^2-(ax+b)^2}{y^3}$$

$$\therefore \quad y^3 \frac{d^2y}{dx^2} = a\,(ax^2+2bx+c) - (ax+b)^2$$

$$= a^2x^2+2abx+ac-a^2x^2-2abx-b^2$$

$$= ac-b^2, \text{ a constant.}$$

Ex. 3. If $= x+\sqrt{x^2-1}$ show that

$$(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$$

Sol. $$\frac{dy}{dx}=1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x$$

$$=\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}$$

$$=\frac{y}{\sqrt{x^2-1}} \qquad \text{(using the given relation)}$$

$$\therefore \quad \sqrt{x^2-1}\frac{dy}{dx}=y \qquad \ldots(1)$$

Differentiating (1) w.r.t. x as an implicit function we get

$$\sqrt{x^2-1}\cdot\frac{d^2y}{dx^2}+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\cdot\frac{dy}{dx}=\frac{dy}{dx}$$

$$\therefore \quad (x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=\sqrt{x^2-1}\frac{dy}{dx}$$

$$\therefore \quad (x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=y \qquad \text{from (1)}$$

$$\therefore \quad (x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$$

EXERCISE

1. Find the second order derivatives of the following functions,

   (*i*) e^{3x} (*ii*) $\log(x+1)$ (*iii*) $\tan x$

   (*iv*) $\sin^{-1}x$, (*v*) $x^2\cos x$

2. If $y = e^{mx}(ax+b)$, prove that

$$\frac{d^2y}{dx^2}-2m\frac{dy}{dx}+m^2y=0$$

3. If $y = e^x(\cos x+\sin x)$ show that

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$$

4. If $y = e^{m\sin^{-1}x}$, show that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$$

5. If $y = (x+\sqrt{x^2+1})^3$, prove that

$$(x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-9y=0$$

ANSWERS

1. (*i*) $9e^{3x}$ (*ii*) $-\frac{1}{(x+1)^2}$ (*iii*) $2\tan x\sec^2 x$

(*iv*) $\frac{x}{(1-x^2)^{3/2}}$ (*v*) $\cos x\,(2-x^2)-4x\sin x$

4

Applications of Derivatives

In the last chapter, we discussed the techniques of obtaining the derivatives of various functions. We shall now consider some simple applications of derivatives in Geometry, Mechanics optimization problems in different fields and the problems of approximations and error analysis. We shall also discuss how a differentiable functions are characterised by their derivatives.

4·1. Approximations and Errors

We know that, if a function f is differentiable at a point a, then its derivative $f'(a)$ at the point a is given by

$$\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right] = f'(a).$$

This means that, when h is *sufficiently* small, the value of the expression $\left[\frac{f(a+h)-f(a)}{h}\right]$ is very close to $f'(a)$ *i.e.* for sufficiently small h, $(h \neq 0)$ $\frac{f(a+h)-f(a)}{h}$ is approximately equal to $f'(a)$. We denote this as $\frac{f(a+h)-f(a)}{h} = f'(a)$ for *sufficiently* small h.

On rewriting the above expression, we get

$$f(a+h) = f(a)+h\,f'(a), \quad \text{where } h \text{ is small.}$$

Thus, we can estimate the value of a differentiable function f at a point $(a+h)$, $h\neq 0$ where h is sufficiently small, if we know the value of the function f at a and also the derivative of f at a. It is observed that for known differentiable functions, the approximate value of the function obtained with the help of above formula does not *differ appreciably* from the actual value for sufficiently small values of h.

Illustrative Examples

Ex. 1. Find an approximate value of $\sqrt{99}$ with the help of derivative.

Sol. Let the function f be defined as $f(x) = \sqrt{x}$.

Choose now $a = 100$ because we can compute the value of $f(a) = f(100)$ without any difficulty and 100 is a convinient number close to 99 which satisfies the requirement.

Now, putting $a = 100$, $h = -1$ in $f(a+h) = f(a)+a f'(a)$, we get

$$\therefore \quad f(100-1) = f(100)+(-1)f'(100).$$

$$\therefore \quad f(99) = f(100)+(-1)\frac{1}{2\sqrt{100}}.$$

$$\therefore \quad \sqrt{99} = \sqrt{100} + \frac{(-1)}{2\times 10}$$

$$\therefore \quad \sqrt{99} = 10-0{\cdot}05$$

$$\therefore \quad \sqrt{99} = 9{\cdot}95$$

$\therefore$ Approximate value of $\sqrt{99}$ is 9·95.

Ex. 2. With the help of derivatives, find an approximate value of x^2+5x^2+3x+2 when $x = 2{\cdot}01$

Sol. Let $f(x) = x^3+5x^2+3x+2.$

Then, $f'(x) = 3x^2+10x+3.$

It is easy to obtain the value of $f(x)$ at $x = 2$ and 2 is closer to 2·01 than any integer.

Putting $a = 2$ and $h = {\cdot}01$ in

$$f(a+h) = f(a)+h f'(a),$$

we get

$$f(2{\cdot}01) = f(2)+({\cdot}01)f'(2).$$

Now, $f(2) = (2)^3+5(2)^2+3(2)+2$

$= 36$

and $f'(2) = 3(2)^2+10(2)+3$

$= 35.$

$\therefore$ We get

$$f(2{\cdot}01) = 36+({\cdot}01)(35)$$

i.e. $f(2{\cdot}01) = 36{\cdot}35.$

Ex. 3. Using derivatives, find approximate values of

(*i*) $\tan (45° 12')$; (*ii*) $(10)^{3.01}$.

Sol. (*i*) Let $f(x) = \tan x$. Then,

$$f'(x) = \sec^2 x.$$

For sufficiently small h, we have

$$f(a+h) = f(a)+h f'(a).$$

Taking $a = \frac{\pi}{4}^c$, $h = 12' = \cdot0035^c$, we get

$$f\left(\frac{\pi}{4}+\cdot0035\right) = f\left(\frac{\pi}{4}\right)+\cdot0035\times f'\left(\frac{\pi}{4}\right).$$

$$\therefore \quad \tan\left(\frac{\pi}{4}+\cdot0035\right) = \tan\left(\frac{\pi}{4}\right)+\cdot0035\times\sec^2\left(\frac{\pi}{4}\right).$$

$$\therefore \quad \tan (45° 12') = 1\cdot0035. \qquad \because \quad \sec^2\frac{\pi}{4} = 2.$$

(*ii*) We know that

$$\log_e 10 = 2\cdot3026.$$

Now, let $f(x) = 10^x$.

then, we have $f'(x) = 10^x \log_e 10$

$\therefore$ $f(3) = 10^3 = 1{,}000$

and $f'(3) = 10^3 \log_e 10$

$= 1{,}000 \times 2\cdot3026 = 2{,}302\cdot6.$

Putting $h = 0\cdot01$ and $a = 3$ *in*

$$f(a+h) = f(a)+hf'(a),$$

we get,

$$f(3\cdot01) = f(3)+0\cdot01\times f'(3).$$

$$\therefore \quad 10^{3.01} = 1{,}000+0\ 01\times 2{,}302\cdot6$$

$$\therefore \quad 10^{3.01} = 1{,}023\cdot026.$$

Ex. 4. The angle of elevation of the top of a tower at a distance of 100 meters from its base is 30°. Find the height of the tower. Compute the approximate error in measuring the height if the actual angle of elevation is 30° 12′.

Sol. Let z be the height of the tower AB and let C be a point at 100 meters from its base. If the angle of elevation of the top of the tower at C is θ, then

$$\frac{z}{100} = \tan\theta.$$

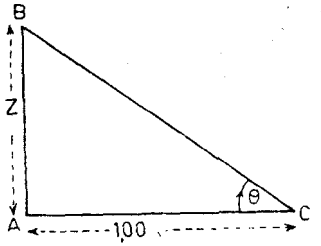

Fig. 4·1.

$\therefore \quad z = 100 \tan\theta.$

Now, when $\theta = 30°$,

$$z = 100 \tan 30° = 100 \times 0{\cdot}5774$$
$$= 57{\cdot}74 \text{ meters.}$$

Put $z = f(\theta)$

$\therefore \quad f(\theta) = 100 \tan(\theta).$

$\therefore \quad f'(\theta) = 100 \sec^2(\theta).$

Now, putting $\theta = \frac{\pi^c}{6}$

and $h = 12' = {\cdot}0035^c$

in $f(\theta+h) = f(\theta) + hf'(\theta),$

we get

$$f\left(\frac{\pi^c}{6} + {\cdot}0035^c\right) = f\left(\frac{\pi^c}{6}\right) + 0{\cdot}0035 \times f'\left(\frac{\pi^c}{6}\right)$$

$$\therefore f\left(\frac{\pi^c}{6} + 0{\cdot}0035^c\right) = 57{\cdot}74 + 0{\cdot}0035 \times \sec^2\left(\frac{\pi^c}{6}\right)$$

$$\therefore f\left(\frac{\pi^c}{6} + 0{\cdot}0035\right) = 57{\cdot}7447 \text{ meters.}$$

$\therefore$ The error in measuring the height of the tower will be (57·7447—57·74) meters or 4·7 cms. approximately.

EXERCISE 4·1

1. Using derivatives, find approximate values of

(*i*) $\sqrt[5]{32{\cdot}044}$; (*ii*) $\sqrt[4]{255}$; (*iii*) $\sqrt[3]{511}$;

(*iv*) $\sqrt{100{\cdot}1}$; (*v*) $\sqrt[3]{997}$.

2. With the help of derivatives, find approximate values of

(*i*) sin (29° 12′); (*ii*) cos (45° 30°);

(*iii*) tan (45° 30′).

3. With the help of derivatives, find an approximate value of

$$x^3-2x^2+3x+2 \qquad \text{when } x=3{\cdot}02.$$

4. If $f(x)=x^4+2x^3+3$, find an approximate value of $f(2{\cdot}01)$.

5. If $f(x)=x^5+x^3+x$, find an approximate value of $f(3{\cdot}2)$.

6. The angle of elevation of the top of a tower from a distance of 100 meters from its base is 45°. Find the height of the tower and compute approximately the error in the measurement of the height if the actual angle of elevation is 44° 40′ (Take $1^c=0{\cdot}0175^c$).

7. The radius of a circle is 7·2 cms. find approximately the error in computing its area if the radius is taken as 7 cms.

8. If $\log_e 10=2{\cdot}3026$, find approximate value of $\log_e 101$.

9. The sides of a triangle ABC are correctly measured but there is an error of h^c in measuring the angle A. Show that the error in calculating the area of the triangle is approximately $h/2\ b\ c \cos A$.

10. The periodic time T of a simple pendulum of length l is given by

$$T=2\pi\sqrt{\frac{l}{g}}$$

where g is the accoleration due to the gravity. If the length of the pendulum is altered by 2%, show that the periodic time of the pendulum will change by approximately 1%.

ANSWERS

1. (*i*) 2·00055 ; (*ii*) 3·995 ; (*iii*) 7·917, (*iv*) 10·005 ; (*v*) 9·99.

2. (*i*) 0 5264 ; (*ii*) 0·70091 ; (*iii*) 1·0175.

3. 20·36.

4. 27·4.

5. 359·6.

6. 1·164 cm.

7. 8·8 cm^2.

8. 4·6152.

4·2. Applications to Geometry

1. Tangents and Normals.

Let $y=f(x)$ be the equation of a smooth curve and let $A\ (a, f(a))$ and $B\ (a+h, f(a+h))$ be two distinct points on the curve.

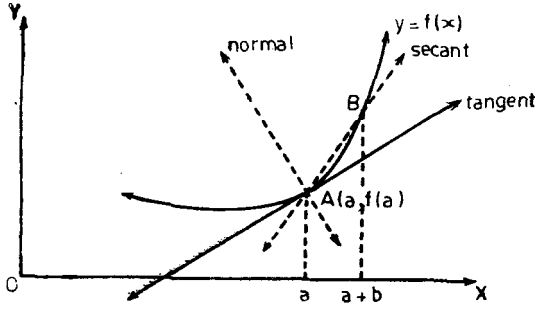

Fig. 4·2.

Then, AB is the secant of the curve through the points A and B and its slope is

$$\frac{f(a+h)-f(a)}{h}.$$

The tangents to the curve $y=f(x)$ at A $(a, f(a))$ is the limiting position of the secant AB as $B\to A$ along the curve. Therefore, the slope of the tangent at A is the limit of the slope of the secant AB as $B\to A$ along the curve.

Now, as $B\to A$ along the curve, $h\to 0$.

$\therefore$ The slope of the tangent to the curve $y=f(x)$ at A $(a, f(a))$

$$= \lim_{h\to 0} [\text{slope of the secant } AB].$$

$$= \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right] = f'(a).$$

Note that $f'(a)=\left(\frac{dy}{dx}\right)_{x=a}$.

$\therefore$ Slope of the tangent to the curve $y=f(x)$ at $x=a$ is

$$\left(\frac{dy}{dx}\right)_{x=a}.$$

2. Subtangents and Subnormals

Let the tangent and the normal to the curve $y=f(x)$ at the point P $(a, f(a))$ on it, meet the axis of X in T and G respectively.

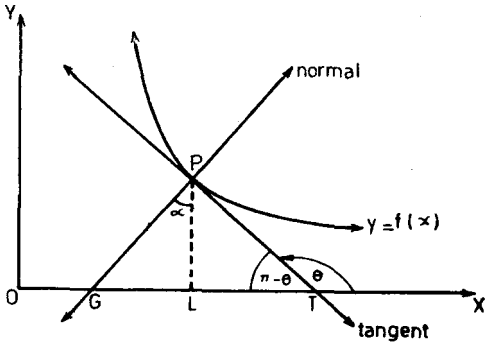

Fig. 4.3

Let L be the foot of the perpendicular from P on the X-axis Then,

(*i*) LT is called *the subtangent* at P,

and (*ii*) LG is called the *subnormal* at P.

Since the slope of the tangent to the curve at P is $f'(a)$ if, θ is the angle of inclination of the tangent PT, then we get

$$\tan\theta = f'(a).$$

Now, from the right angled triangle PLT, we get

$$\frac{l(PL)}{l(LT)} = |\tan(\pi-\theta)| = |\tan\theta| = |f'(a)|,$$

$$\therefore \quad l(LT) = \frac{l(PL)}{|f'(a)|} = \left|\frac{f(a)}{f'(a)}\right|$$

$\therefore$ The length of the subtangent LT is

$$\left|\frac{f(a)}{f'(a)}\right| \text{ or } \left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|_{x=a}$$

Similarly from the right angled triangle PGL, we get (Fig. 4·3.)

$$\tan\alpha = \frac{l(GL)}{l(PL)} = |\tan(\pi-\theta)| = |\tan\theta| = |f'(a)|.$$

$$\therefore \quad l(GL) = l(PL)\,|f'(a)| = |f(a).f'(a)| = \left|y\left(\frac{dy}{dx}\right)\right|_{x=a}.$$

In general, the length of the subnormal to the curve at $P(x, y)$ on it is given by $\left|y\frac{dy}{dx}\right|$.

3. Angle between two curves.

Let two curves $y=f(x)$ and $y=g(x)$ intersect at a point P. Then, the *angle between the two curves* is defined to be the acute angle between the tangents to the curves at P.

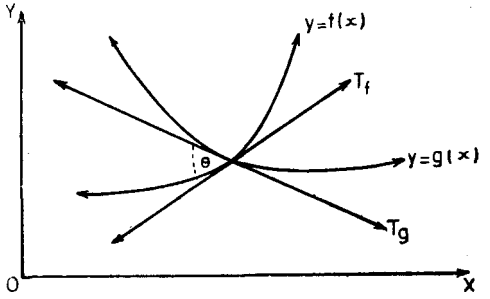

Fig. 4·4.

Now, suppose that abscissa of P is a. Then the slope of the tangent to the curve $y=f(a)$ at P is $f'(a)$ and the slope of the tangent to the curve $y=g(x)$ at P is $g'(a)$.

Then, if α is the acute angle between these two tangents at P, then

$$\tan \alpha = \left| \frac{f'(a)-g'(a)}{1+f'(a).g'(a)} \right|$$

$\therefore \tan^{-1} \left| \frac{f'(a)-g'(a)}{1+f'(a).g'(a)} \right|$ is the angle between the curves $y=f(x)$ and $y=g(x)$ at their point of intersection $P\ (a, f(a))$ or $P\ (a, g(a))$.

Illustrative Examples.

Ex. 7. Find the equations of the tangents and normals to the curve $y^2=x(5a-x)$ at the points on it where $x=a$.

Sol. Putting $x=a$ in $y^2=x(5a-x)$,

we get

$$y^2=a\ (5a-a)=4a^2$$

$$\therefore \quad y=\pm 2a.$$

$\therefore$ We have to find the equations of the tangents and normals to the curve at the points $(a, 2a)$ and $(a, -2a)$ on it.

Differentiating $y^2=x(5a-x)$, we get,

$$2y\frac{dy}{dx}=5a-2x$$

or

$$\frac{dy}{dx}=\frac{5a-2x}{2y}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(a,\ 2a)}=\frac{5a-2a}{2(2a)}=\frac{3}{4}$$

and

$$\left(\frac{dy}{dx}\right)_{(a,\ -2a)}=\frac{5a-2a}{2(-2a)}=-\frac{3}{4},$$

$\therefore$ The slopes of the tangent and normal to the curve at $(a, 2a)$ are respectively $\frac{3}{4}$ and $-\frac{4}{3}$.

$\therefore$ The equations of the tangents and normal at $(a, 2a)$ are respectively.

$$y-2a=\frac{3}{4}\ (x-a)$$

and

$$y-2a=-\frac{4}{3}\ (x-a)$$

Similarly, the slopes of the tangent and normal to the curve at $(a,-2a)$ are respectively $-\frac{3}{4}$ and $\frac{4}{3}$, and their respective equations

are $$y+2a=-\frac{3}{4}(x-a)$$

and $$y+2a=\frac{4}{3}(x-a).$$

Ex. 2. Find the length of the subtangent and subnormal of the rectangular hyperbola $x^2-y^2=a^2$ at the point $(\sqrt{2a}, a)$ on it.

Sol. Differentiating $x^2-y^2=a^2$ with respect to x, we get

$$2x-2y\frac{dy}{dx}=0.$$

$$\therefore \quad \frac{dy}{dx}=\frac{x}{y}$$

$\therefore$ The length of the subtangent at $(\sqrt{2a}, a)$ is

$$\left|\frac{y}{\left(\frac{dy}{dx}\right)}\right|_{(\sqrt{2a},a)}=\left|\frac{y}{\left(\frac{x}{y}\right)}\right|_{(\sqrt{2a},a)}$$

$$=\left|\frac{y^2}{x}\right|_{(\sqrt{2a},a)}=\left|\frac{a^2}{\sqrt{2a}}\right|$$

$$=\frac{a\sqrt{a}}{\sqrt{2}}.$$

Similarly, the length of the subnormal at $(\sqrt{2a}, a)$ is

$$\left|\left(y\frac{dy}{dx}\right)\right|_{(\sqrt{2a},a)}=\left|y\left(\frac{x}{y}\right)\right|_{(\sqrt{2a},a}=\sqrt{2a}.$$

Ex. 3. Find the angle at which the curves $y^2=4x$ and $xy=16$ intersect.

Sol. Solving $y^2=4x$ and $xy=16$ to get the point of intersection, we get

$$y^2=4x=4\left(\frac{16}{y}\right)$$

$$\therefore \quad y^3=64 \quad \text{or} \quad y=4$$

and consequently $x=4$,

$\therefore$ (4, 4) is the point of intersection of the curves.

We now find the slopes of the tangents to the curves at (4, 4).

Differentiating $y^2=4x$, we get

$$\frac{dy}{dx}=\frac{2}{y}.$$

$\therefore$ Slope of the tangent to the curve $y^2=4x$ at (4, 4) is

$$\left(\frac{dy}{dx}\right)_{(4,\ 4)}=\left(\frac{2}{y}\right)_{(4,\ 4)}$$

$$=\frac{2}{4}=\frac{1}{2}.$$

Similarly, differentiating $xy=16$, we get,

$$\frac{dy}{dx}=-\frac{y}{x}=-\frac{16}{y^2}.$$

$\therefore$ The slope of the tangent to this curve at (4, 4) is

$$\left(\frac{dy}{dx}\right)_{(4,\ 4)}=-\frac{16}{16}=-1,$$

$\therefore$ The slopes of the tangents to the curves at (4, 4) are $\frac{1}{2}$ and -1,

$\therefore$ The angle between the curves is,

$$\tan^{-1}\left|\frac{-1-\frac{1}{2}}{1-(-1)\frac{1}{2}}\right|=\tan^{-1}(1)=45^\circ$$

The curves intersect at an angle of 45°.

EXERCISE 4·2.

1. Find the equation of the tangent and normal to the curve $y=x^3-3x^2-x+5$ at (3, 2).

2. Find the equation of the tangents to the circle $x^2+y^2=5^2$ which are parallel to the line $2x+3y=5$.

3. Find the equation of the normal to the hyperbola $4x^2-9y^2=36$ which is perpendicular to the line $x+2y=3$.

4. Find the equation of the tangent and normal to the curve $y=\frac{3}{x+4}$ at the point where $x=2$.

5. Find the equations of the tangents and normals to the curve $y^2(a+x)=x^2(3a-x)$ at the points where $x=a$.

6. Find the equations of the tangents to the curve $y=x^2+3y^2-9x+7$ which are parallel to the X-axis.

7. Show that the curves $y=3x^2-8$ and $y=x^3-4$ have a common tangent at the point where $x=2$. Find its equation.

8. Find the equations of the tangents to the curve $y=x^2+4x+16$ which pass through the origin.

9. Show that the sum of the intercepts made by a tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ on the co-ordinate axes is constant.

10. Show that the portion of the tangent at any point of the rectangular hyperbola $xy=a^2$, intercepted between the coordinate axes is bisected at the point of contact.

11. Find the lengths of the subtangent and subnormal of

(*i*) $x^2+y^2=a^2$ at the point $(a\cos\theta, a\sin\theta)$,

(*ii*) $xy=a^2$ at the point $\left(2a, \frac{a}{2}\right)$ and

(*iii*) $x^{2/3}+y^{2/3}=a^{2/3}$ at the point $(a\cos^3\theta, a\sin^3\theta)$.

12. Show that for the curve $y=3^{x+2}$, the subtangent is of constant length and for the curve $y=\sqrt{x+3}$, the subnormal is of constant length.

13. For the curve $y=a^x$, obtain the X-co-ordinate of the point at which the lengths of the subtangent and subnormal are equal.

14. Show that the curves $\frac{x^2}{2}+\frac{y^2}{4}=1$ and $y^2=16x$ intersect at right angles.

15. Show that the angles of intersection between the curves $x^2=4ay$ and $2y^2=ax$ are $\frac{\pi}{2}$ and $\tan^{-1}\left(\frac{3}{5}\right)$.

16. Find the angle of intersection of the curves $x^2+y^2=8$ and $x^2-y^2=4$.

17. Find the angles at which the curves $y=4ax$ and $x^2=4ay$ intersect.

18. Show that the circles $x^2+y^2-4x=0$ and $x^2+y^2-8y=0$ intersect at right angles.

ANSWERS

1. $8x-y-22=0$; $x+8y+9=0$;

2. $2x+3y\pm26=0$;

3. $x+2y=\frac{\pm17}{8}\sqrt{2}$;

4. $y=2x-6$;

5. $2ax-y=3a$; $2ax+y=3a$;

6. $y=3$, $y=37$;

7. $12x-y-20=0$;

8. $y=12x$; $y+4x=0$;

9. (*i*) $a\tan\theta\sec\theta$; $a\cos\theta$;

(*ii*) $2a$; $\frac{9}{8}$;

(*iii*) $a\sin^2\theta\cos\theta$; $a\tan\theta\sin^3\theta$;

16. $60°$; **17.** $\frac{\pi}{2}$; $\tan^{-1}\left(\frac{3}{4}\right)$.

43. Applications to Mechanics

We know that if y is a function of the variable x, then $\left(\frac{dy}{dx}\right)$ stands for the rate of change of y with respect to x. But when

physical meanings are attached to y and x, then in different branches, $\left(\frac{dy}{dx}\right)$ has different names. For example, if y represents displacement and x represents time then $\left(\frac{dy}{dx}\right)$ is called *the velocity*; if y is the charge and x represents time then $\left(\frac{dy}{dx}\right)$ is called the current ; if y is the velocity and x is the time, then $\left(\frac{dy}{dx}\right)$ is called *the acceleration* etc.

In mechanics we are primarily concerned with the velocity and acceleration of a moving object. We shall, henceforth, represent the displacement by x, the time by t, the velocity by v and the acceleration by f.

Clearly,

Velocity v = the time rate of change of the displacement.

$$= \frac{dx}{dt}$$

and, acceleration f = the time rate of change of the velocity.

$$= \frac{dv}{dt}.$$

We can use *the chain rule* for the derivatives and write,

$$f = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \left(\frac{dv}{dx}\right) v = v \frac{dv}{dx}.$$

Illustrative Examples

Ex. 1. A particle moving along a straight line is at a distance x from the origin at time t, where $x = t^3 - 3t^2 + 3t + 5$.

Find (i) the time when the velocity is zero and (ii) the distance when the acceleration is zero.

Sol. $x = t^3 - 3t^2 + 3t + 5$

$$\therefore \quad v = \frac{dx}{dt} = 3t^2 - 6t + 3$$

and $f = \frac{dv}{dt} = 6t - 6.$

$\therefore$ $v = 0$ when $3t^2 - 6t + 3 = 0$

i.e., $3(t-1)^2 = 0$

$\therefore$ $v = 0$ when $t = 1$.

Also, $f = 0$ when $6t - 6 = 0$.

i.e., $t = 1$.

Now, at $t = 1$, $x = 1^3 - 3(1)^2 + 3(1) + 5 = 6$.

$\therefore$ When acceleration is zero, $x = 6$.

Ex 2. A particle moves along a straight line under the law, $x = ae^{-t} + be^{t}$. (*i*) Show that the acceleration $f = x$; (*ii*) Find a and b if the particle starts from the origin with velocity 4.

Sol. (*i*) $x = ae^{-t} + be^{t}$

$$\therefore \quad v = \frac{dx}{dt} = -ae^{-t} + be^{t}$$

$$\therefore \quad f = \frac{dv}{dt} = ae^{-t} + be^{-t} = x.$$

(*ii*) When $t = 0$, $x = 0$ and $v = 4$.

$\therefore$ $ae^{0} + be^{0} = 0$ and $-ae^{0} + be^{0} = 4$.

$\therefore$ $a + b = 0$ and $-a + b = 4$.

$\therefore$ $b = 2$ and $a = -2$.

$\therefore$ $x = -2e^{-t} + 2e^{t}$.

Ex. 3. For a particle moving along a straight line, $v = \sqrt{1 + x^2}$ where v is the velocity at time t and x is the distance of the particle from the origin at time t. Show that acceleration $f = x$ and find the acceleration and the distance of the particle when $v = \sqrt{26}$.

Sol. $v = \sqrt{1 + x^2}$

$$\therefore \quad \frac{dv}{dx} = \tfrac{1}{2}(1 + x^2)^{-\frac{1}{2}}(2x)$$

$$= \frac{x}{\sqrt{1 + x^2}}$$

$$\therefore \quad f = v\frac{dv}{dx}$$

$$= \sqrt{1 + x^2} \cdot \frac{x}{\sqrt{1 + x^2}}$$

$$= x.$$

Now, when $v = \sqrt{26}$, $x^2 = v^2 - 1$.

$$\therefore \quad x^2 = 26 - 1.$$

$$\therefore \quad x^2 = 25.$$

$\therefore$ $x = \pm 5$ when $v = \sqrt{26}$.

$\therefore$ $f = x = \pm 5$ when $v = \sqrt{26}$.

Ex. 4. A particle moves along a straight line OX under the law $x = a \cos \mu t$ where x is the distance of the particle from the

origin at time t. Show that the acceleration at time t is $-\mu^2 x$ and obtain the velocity of the particle when $x = \frac{a}{2}$.

Sol. $x = a \cos \mu t$.

$\therefore \quad v = \frac{dx}{dt} = -\mu a \sin \mu t.$

$\therefore \quad f = \frac{dv}{dt} = -\mu^2 a \cos \mu t.$

$\therefore \quad a = -\mu^2 x \qquad (\because\ x = a \cos \mu t).$

Now, $v = -\mu a \sin \mu t$

$= -\mu \sqrt{a^2 - x^2} \qquad (\because\ a^2 \sin^2 \mu t = a^2 - a^2 \cos^2 \mu t = (a^2 - x^2).$

$\therefore$ When $x = \frac{a}{2}$,

$$v = -\mu \sqrt{\frac{a^2 - a^2}{4}}$$

$$= -\mu a \frac{\sqrt{3}}{2}.$$

$\therefore$ When $x = \frac{a}{2}$, $v = \frac{-\mu a \sqrt{3}}{2}$.

EXERCISE 4·3.

1. A body moves along a straight line according to the law $x = \frac{1}{2} t^2 - 2t$. Find the time when the velocity is 4 units and also the velocity when time $t = 3$.

2. A particle is moving in a straight line and its distance x cm from a fited point A on the line at the time t seconds is given by $x = \sqrt{1 + t^2}$. Show that the acceleration of the particle is x^2 cm s./sec^2. at the time t.

3. A particle moves under the law $v^2 = k\sqrt{x}$, where v is its velocity and x, the distance from the origin at time t. Find its acceleration.

(*i*) when it is at a distance of 100 cms. and

(*ii*) when its velocity is 4 cms/sec.

4. A car starts from rest and its velocity v meters per second is given in terms of x the distance travelled in meters, by the equation.

$$V = \frac{3x}{20} - \frac{x\sqrt{x}}{200}$$

until the maximum velocity is reached. Show that the acceration when it has travelled 16 meters is approximately 25 cms./sec^2.

5. A particle moves along the axis of X according to the law $x = ae^{-1/2t} + be^t$, x being its distance from the origin at time t. Find a and b if it starts from the origin with a velocity of 15 units.

6. A bullet is shot horizontally and its distance s ft, at time t seconds is given by, $s = 1200\ t - 16\ t^2$. Find the velocity with which the bullet is shot. Also find the distance covered and the time required when it comes to rest.

7. A particle moves along a straight line according to the law $x=256\ t-16\ t^2$, x being its distance from the origin at time t. Find,

(*i*) the time and distance when its velocity is zero.

(*ii*) the time when $x=448$

(*iii*) the velocity with which it returns to the origin.

ANSWERS

1. 2 ; $\frac{23}{2}$ **3.** $\frac{k}{40}$; $\frac{k^2}{64}$; **5.** $a=10$; $b=10$; **6.** $t=40$; $s=25{,}600$; **7.** (*i*) 8 ; 1024 ; (*ii*) 2,14 ; (*iii*) 256.

4·4. Derivative as a rate measurer.

In different branches of science, we deal with entities which are constantly changing ; and very often we have to tackle the problem of obtaining the rate of change entity with respect to the second whenever the first depends on the second. Now, if x and y are two entities and further if y depends on x, then the rate of change of y with respect to x is precisely $\left(\frac{dy}{dx}\right)$. However, in most of the physical situations, we are going to deal with the entities which are time dependent, and hence we shall be primarily concerned with obtaining their derivatives with respect to time t. In such problems, it is important to express different entities involved in terms of one-another to arrive at the required result. We shall consider a few examples to expose the techniques.

Illustrative Examples.

Ex. 1. A circular disc of radius 2 cms. is being heated. Due to expansion, its radius increases at the rate of 0·02 cms./sec; Find the rate at which its area is increasing when the radius is 2·1 cms.

Sol. Let r be the radius of the disc, at time t. Then its area A is given by, $A=\pi r^2$

$$\therefore \quad \frac{dr}{dt}=\pi 2r\frac{dr}{dt}=2\pi r\frac{dr}{dt} \ .$$

Now the radius increases at the rate of 0·02 cms./sec.

$$\therefore \quad \frac{dr}{dt}=0{\cdot}02 \text{ cms./sec.}$$

$$\therefore \quad \frac{dA}{dt}=2\pi r\times(0{\cdot}02)=\cdot 04\pi r \text{ cms}^2\text{./sec.}$$

$\therefore$ When $r=2{\cdot}1$ cms., we have

$$\frac{dA}{dt}=2\pi\times 2{\cdot}1\times\cdot 02$$

$$=2\times\frac{22}{7}\times 2{\cdot}1\times{\cdot}02$$

$$=0{\cdot}264 \text{ cms.}^2/\text{sec.}$$

The area is increasing at the rate of ·264 cm.^{2} per sec. when $r=2{\cdot}1$ cms.

Ex. 2. A man is moving away from a tower 85 ft, high at a speed of 4 ft./sec. Find the rate at which his angle of elevation of the top of the tower is changing, when he is at a distance of 60 ft. from the foot of the tower.

Assume the eye-level of man to be 5 ft·

Sol In Fig. 4·5. B is at the eye level of the man, OP is the tower and $\angle PBC=\theta$ is the angle of elevation of the top of the top of the tower. For the right angled triangle PBC, we have.

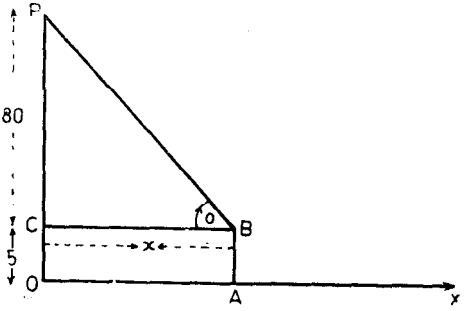

Fig. 4·5

$$\tan\theta=\frac{l(PC)}{l(BC)}.$$

$$\therefore \quad \tan\theta=\frac{80}{x}$$

$$\therefore \quad \sec^2\theta\frac{d\theta}{dt}=\frac{-80}{x^2}\cdot\frac{dx}{dt}$$

Now the man is moving away from the foot of the tower with a speed of 4 ft./sec.

$$\therefore \quad \frac{dx}{dt}=4$$

$$\therefore \quad \frac{d\theta}{dt}=\frac{-80}{x^2}\times 4\times\cos^2\theta$$

$$=\frac{-320}{x^2}\cdot\cos^2\theta$$

But, when $x=60$, $\cos\theta=\frac{60}{100}=\frac{3}{5}$

$\therefore$ When $x=60$,

$$\frac{d\theta}{dt}=\frac{-320}{(60)^2}\cdot\left(\frac{3}{5}\right)^2$$

$$=-0{\cdot}0032 \text{ radians/sec.}$$

$\therefore$ When the man is at a distance if 60 ft. from the foot of the tower, his angle of elevation of the top of the tower is *decreasing* at the rate of 0·0032 radians/sec. $\left(\because \quad \frac{d\theta}{dt}<0\right)$

Ex. 3. A rectangular swimming tank is 25 ft. by 40 ft. Water is pumped into the tank at the rate of 500 cub. ft./min. Find the rate at which the level of the water in the tank is rising.

Sol. Let z=depth of water in the tank at time t.

$\therefore$ The volume v of the water in the tank at time t will be,

$v=25\times40\times z=1{,}000\ z.$

$$\frac{dv}{dt}=1{,}000\ \frac{dz}{dt}, \qquad \ldots(1)$$

Since the water is being pumped into the tank at the rate of 500 cub, ft./min., we have.

$$\frac{dv}{dt}=500. \qquad \ldots(2)$$

$\therefore$ Equating (1) and (2), we get

$$500=1000\ \frac{dz}{dt}.$$

$$\therefore \quad \frac{dz}{dt}=0{\cdot}5$$

$\therefore$ The level of the water in the tank is rising at the rate of 0·5 ft./min.

EXERCISE 4·4.

1. The radius of a soap bubble is increasing at the rate of 0 01 cms / sec. Find the rate at which its volume is increasing when its diameter is 7 cms.

2. A man 6ft. tall is walking away from a lamp post 18 ft. high at a speed of 3ft./sec- Find the rate at which his shadow is increasing and also find, the rate at which the tip of his shadow is moving.

3. A metal sheet is in the shape of an equilateral triangle of side $4\sqrt{3}$ cms. As it is heated gradually, the radius of its circumcircle increases? at the rate of 0·01 cms./sec. Find the rate at which its area is increasing when its radius is 4·2 cms.

4. A cylindrical water reservoir has a base of area 5 sq. meters. Water is being poured into it at the rate of v cub. meters/sec. Find the rate at which the level of water is rising in the reservoir.

5. A storage tank is in the shape of a cone with vertex downwards. The radius of the base is 20 meters. and its vertical height is 40 meters. Oli is being pumped into the tank at the rate of 100 cub. meters/min. Find the rate at which the level of oil in the tank is rising when its inside depth is 10 meters.

6. Water escapes at the rate of 8 cub, meters/min. through the vertex of an inverted conical container whose height is 12 meters and whose base is a circle of diameter 8 meters. Find the rate at which the level of the water in the container is sinking when the water inside is 6 meter deep.

7. The volume of a sphere is increasing at the rate of 100 cub. cms./sec. Find the rate at which its surface area is increasiug when its radius is 20 cms.

8. A stone is dropped into a still pond of water. It sends out concentric circular waves of water which move at the rate of 4 cms./sec. Find the rate of which the disturbed area is increasing when the radius of the largest wavering is 3·5 meters.

ANSWERS

1. 6·16 cms./sec. ;
2. 1·5 ft./sec., 4'5 ft./sec.,
3. 0·11 cms^2/sec.,
4. v/s ;
5. $\frac{4}{\pi}$ meters/min..
6. $\frac{2}{\pi}$ meters/min.,
7. 10 cm^2./sec.,
8. ·22 sq. meters/sec.

4·5. Meaning of the sign of derivative.

Let $y=f(x)$ be a differentiable function. Then as remarked in the earlier chapter, the function f is continuous and therefore its graph is a continuous curve. Now $\left(\frac{dy}{dx}\right)$ is a measure of rate of change of $y=f(x)$ with respect to x and hence whenever $\left(\frac{dy}{dx}\right)$ is positive, y increases with x and whenever $\left(\frac{dy}{dx}\right)$ is negative, y decreases as x increases. In this section, we shall discuss this property of a function in more detail.

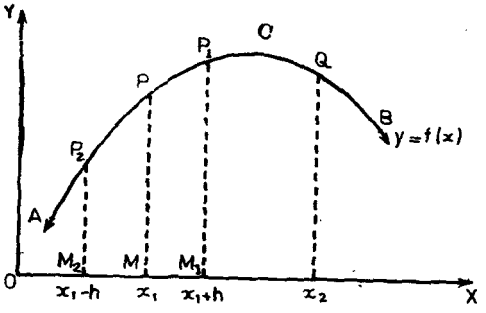

Fig. 4.6.

Suppose that the curve ACB (Fig. 4·6) is the graph of the function $y=f(x)$. The curve is rising from A to C and falling down from C to B. This means, all points (x, y) of the curve between A to C are such that as x increases, y also increases. Similarly, for all points (x, y) on the curve between C and B, y decreases, with increase in the value of x.

Let P_2M_2, PM and P_1M_1 be the ordinates of the points P_2, P. P_1 respectively which are between the points A and C of the curve. Then

$$l(P_1M_1)=f(x+h),\ l(PM)=f(x_1) \text{ and}$$
$$l(P_2M_2)=f(x_1-h).$$

Clearly, $f(x_1-h)<f(x)<f(x_1+h)$. The above inequality is true as long as P_2 and P_1 remain on the left of the point C. This means, the inequality in true for all points P between A and C.

i.e. $$f(x_1-h)<f(x_1)<f(x_1+h)$$

for all sufficiently small $h>0$. We say, in this case that the function f *is increasing* at x_1 (at the point P),

Similarly, we say that the function f is *decreasing* at x_2 (at the point Q) if

$$f((x_2-h)> f(x_2)>f(x_2+h)$$

for all sufficiently small $h>0$.

It can be proved that if $f'(x_1)>0$ then the function f is increasing at x_1 in the above sense and similarly, if $f'(x_2)<0$ then the function f is decreasing at x_2.

i.e. If $f'(x_1)>0$, then for all sufficiently small $x>0$

$$f(x-h)<f(x_1)<f(x_1+h)$$

and $\therefore$ if $f'(x_2)<0$ then, for all sufficiently small $x>0$,

$$f(x_2-h)>f(x_2)>f(x_2+h).$$

Remarks.

1. If for a function f, $f'(x)>0$ for all x in $[a, b]$, then f is increasing throughout the interval $[a, b)$ and graph of the function f is rising on that interval.

2. If for a function $f'(x)<0$ for all x in $[a, b)$, then f is decreasing throughout the interval $[a, b]$ and the graph of the function f is falling on that interval.

3. If f is increasing on $[a, b]$, *it does not* imply that $f'(x)>0$ for all x in $[a, b]$, for example, the function $f(x)=x^3$ is increasing on $[-1, 1)$ but $f'(0)=0$. Similarly, if f is decreasing on $[a, b]$ it *does not* imply that $f'(x)<0$ for all x in $[a, b]$.

Illustrative Examples.

Ex. 1. Show that the function f, given by

$$f(x)=x^2-4x+5,$$

is increasing at $x=4$ and decreasing at $x=1$.

Sol. $f(x)=x^2-4x+5$

$\therefore$ $f'(x)=2x-4$

$\therefore$ $f'(4)=2(4)-4=4>0$ and

$f'(1)=2(1)-4=-2<0.$

$\because$ f is increasing at $x=4$ and decreasing at $x=1$.

Ex. 2. Prove that, if $f(x)=2^x$ and $g(x)=\frac{1}{5^x}$ then f is increasing and g is decreasing everywhere on the set R.

Sol. (*i*) $f(x)=2^x$

$\therefore$ $f'(x)=2^x \log_e 2.$

Now $2^x>0$ for all x and $\log_e 2>0.$

$\therefore$ $f'(x)>0$ for all x in R.

$\therefore$ f is increasing on R.

(*ii*) Now, we have

$$g(x)=\frac{1}{5^x}=\left(\frac{1}{5}\right)^x,$$

$\therefore$ $$g'(x)=\left(\frac{1}{5}\right)^x \log_e\left(\frac{1}{5}\right),$$

But $\left(\frac{1}{5}\right)^x>0$ for all real x and

$$\log_e\left(\frac{1}{5}\right)<0.$$

$\therefore$ $g'(x)<0$ for all real x.

$\therefore$ g is decreasing for all x in R.

Ex. 3. Show that

(*i*) $(1-x)e^x<1$ for all $x>0$.

(*ii*) $\log_e(1+x)>x-\frac{x^2}{2}$ for all $x>0$.

Sol. (*i*) Let $f(x)=1-(1-x)e^x$. It is enough if we show that $f'(x)>0$ for all $x>0$.

We have,

$$f(0)=1-(1-0)e^0=0.$$

and $$f(x)=-(1-x)e^x+e^x=xe^x.$$

$\therefore$ $f'(x)=x\,e^x>0$ for all $x>0$

$\therefore$ f is increasing function of x for all $x>0$ and $f(0)=0$.

Also, f is continuous at $x=0$.

$\therefore$ $f(x)>0$ for all $x>0$.

$\therefore$ $1-(1-x)e^x>0$ for all $x>0$.

$\therefore$ $(1-x)e^x<1$ for all $x>0$.

(*ii*) We have to show that $\log(1+x)>x-\frac{x^2}{2}$ for all $x>0$.

Let $f(x)=\log(1+x)-x+\frac{x^2}{2}$.

Clearly $f(0)=0$.

Now, $f'(x)=\frac{1}{1+x}-1+x=\frac{1+x^2-1}{1+x}$

$=\frac{x^2}{1+x}$.

$\therefore$ $f'(x)>0$ for all $x>0$.

$\therefore$ f is an increasing function for $x>0$ and $f(0)=0$.

Also, f is continuous at 0.

$\therefore$ $f(x)>0$ for all $x>0$.

$\therefore$ $\log_e(1+x)-x+\frac{x^2}{2}>0$ for all $x>0$.

$\therefore$ $\log_e(1+x)>x-\frac{x^2}{2}$ for all $x>0$.

EXERCISE 4·5.

1. Examine for which values of x the following functions are increasing and for which values of x they are decreasing.

(*i*) x^2-3x+2; (*ii*) x^3-6x^2;

(*iii*) x^3-3x^2+3x+1; (*iv*) x^3-3x;

(*v*) $2+5x-x^2$; (*vi*) $4-3x$.

2. Show that $\sin x$ is an increasing function on $\left(0, \frac{\pi}{2}\right)$ and a decreasing function on $\left(\frac{\pi}{2}, \pi\right)$.

3. Show that $\log_e x$ is an increasing function of x for all $x\geqslant 1$.

4. Show that a^x is an increasing function if $a>1$ and decreasing function if $0<a<1$, for all values of x.

5. Show that $\tan^{-1} x$ is an increasing function of x for all x.

6. Establish the following inequalities with the help of the results discussed in this section.

(*i*) $1-\frac{x^2}{2}<\cos x\leqslant 1$ for $x>0$.

(*ii*) $x - \frac{x^3}{3!} < \sin x < x$ for all $x > 0$.

(*iii*) $x - \frac{x^2}{2} < \log_e (1+x) < x$ for all $x > 0$.

(*iv*) $\frac{x}{1+x} < \log_e (1+x) < x$ for all $x > 0$.

ANSWERS

1. (*i*) Increasing for $x > \frac{3}{2}$; decreasing for $x < \frac{3}{2}$; (*ii*) increasing for $x < 0$ and $x > 4$; decreasing for $0 < x < 4$; (*iii*) increasing; (*iv*) increasing for $x < 0$ and $x > 1$; decreasing for $0 < x < 1$; (*v*) increasing for $x < \frac{5}{2}$; decreasing for $x > \frac{5}{2}$; (*vi*) decreasing.

4·6. Maxima and minima

Let the graph of a function f be as shown in fig. 4·7. The graph has what may be called peak points at A, C, E and G, and trough points at B, D and F. Consider a peak-point such as A, with co-ordinates $(a, f(a))$. Observe that the ordinate of A which is equal to $f(a)$ is greater than the ordinates of all points in its immediate neighbourhood. This means that the value of the function at $x = a$ is greater than the value of the function at every point in the immediate neighbourhood of $x = a$. Note that we are considering points only in the immediate neighbourhood of $x = a$. Mathematically this may be put as $f(a) > f(x)$ for all x in $(a-h, a+h)$, for sufficiently small $h > 0$.

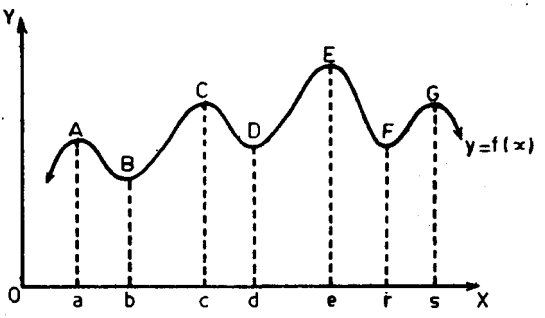

Fig. 4·7.

In this case we say that f has a maximum at (the peak point) $x = a$.

$\therefore$ $f(a)$ is a maximum value of f.

$\therefore$ $f(x)$ is said to be a maximum at $x = p$ if $f(p) > f(x)$ for all x in $(p-h, p+h)$ for sufficiently small $h > 0$.

Thus at each peak point the function f has a maximum.

Similarly, consider a t-rough point such as B with co-ordinates $(b, f(b))$.

Observe that the ordinate *at* B which is equal to $f(b)$ is less than the ordinates of all points in the immediate neighbourhood of B. This means that the value of the function at $x = b$ is less than its value at every other point in the immediate neighbourhood of $x = b$. Mathematically this is put as, $f(b) < f(x)$ for all x in $(b-h, b+h)$ for sufficiently small $h > 0$.

In this case, we say that f has a *minimum* at (the trough point) $x = b$ or $f(x)$ is a *minimum* at $x = b$.

$\therefore$ $f(x)$ is said to be a minimum at $x = q$ if $f(q) < f(x)$ for all x in $(q-h, q+h)$ for sufficiently small $h > 0$.

Thus at each trough point the function f has a minimum.

One *must not* confuse the ideas of maximum and minimum defined above with the usual meaning of maximum (greatest) and minimum (least). The maximum and minimum defined above can be respectively called *a local maximum* and *a local minimum* in the usual sense. A function may have more than one maximum and more than one minimum as illustrated in the Fig. 4·7.

Conditions for Maxima and Minima

Let f be a twice differentiable function. Then it can be shown that

(*i*) f has a maximum at $x = a$ if and

$$\begin{cases} f'(a) = 0 \\ f''(a) < 0 \end{cases}$$

(*ii*) f has a minimum at $x = a$ if

$$\begin{cases} f'(a) = 0 \\ f''(a) > 0. \end{cases}$$

Note. If $f'(a) = 0 = f''(a)$, then no conclusion can be drawn. The function f may or may not have a maximum or a minimum at $x = a$.

Illustrative Examples

Ex. 1. Investigate maximum and minimum of

$$f(x) = 10x^3 - 15x^2 + 10.$$

Sol. We have, $f(x) = 10x^3 - 15x^2 + 10$

$\therefore$ $f'(x) = 30x^2 - 30x$

$= 30x(x-1)$.

$\therefore$ $f'(x) = 0$ implies $30x(x-1) = 0$.

$\therefore$ $f'(x) = 0$ when $x = 0$ or $x = 1$.

Now, $f''(x) = 60x - 30 = 30(2x - 1)$.

Clearly, $f''(0) = 30\,(0-1) = -30 < 0$

and $f''(1) = 30\,(2-1) = 30 > 0$.

Since $f'(0) = 0$ and $f''(0) < 0$, the function f has a maximum at $x = 0$. The maximum value is $f(0) = 10$.

Similarly, since $f'(1) = 0$ and $f''(1) > 0$, the function f has a minimum at $x = 1$. The minimum value is $f(1) = 5$

Ex. 2. Prove that of all the rectangles that can be inscribed in a circle, the square has the greatest area.

Sol. Let $ABCD$ be a rectangle inscribed in a circle of radius a. The diagonal BD is a diameter of the circle.

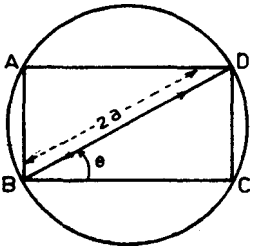

Fig. 4.8.

$\therefore \quad l\,(BD) = 2a.$

Let $m\,\angle ADB = \theta$. $\quad \therefore \quad 0 < \theta < \pi.$

$\therefore$ From the right angled triangle ABD, we have

$$l\,(AB) = 2\,a\,\sin\,\theta$$

and $$l\,(AD) = 2\,a\,\cos\,\theta.$$

$\therefore$ S, the area of the rectangle, is given by

$$S = l\,(AB) \times l\,(AD)$$
$$= 2\,a\,\sin\,\theta \times 2\,a\,\cos\,\theta$$
$$= 2\,a^2\,\sin\,2\,\theta.$$

$S = 2\,a^2\,\sin\,2\,\theta$ is a function of θ.

$\therefore$ S has a maximum if

$$\frac{dS}{d\theta} = 0 \text{ and } \frac{d^2S}{d\theta^2} < 0$$

Now, $\dfrac{dS}{d\theta} = 4a^2\,\cos\,2\,\theta = 0$ implies

$= \cos\,2\,\theta = 0$, But $0 < 2\,\theta < \pi$.

$\therefore \quad 2\,\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{\pi}{4}$.

Also, $\left(\frac{d^2S}{d\theta^2}\right)_{\theta=\pi/4} = (-8\ a^2\ sin\ 2\ \theta)_{\theta=\pi/4} = -8\ a^2 < 0.$

$\therefore \quad \theta = \frac{\pi}{4}\ , \frac{dS}{d\theta} = 0$ and $\frac{d^2S}{d\theta^2} < 0.$

$\therefore$ S is maximum when $\theta = \frac{\pi}{4}$. With this $ABCD$ becomes a square.

Ex. 3. Find the dimensions of a cylinder of volume 16π so that its total surface is minimum.

Sol. Let r be the radius of the base and h the altitude of the cylinder. If S is its total surface, then, $S = 2\pi r^2 + 2\pi rh$,

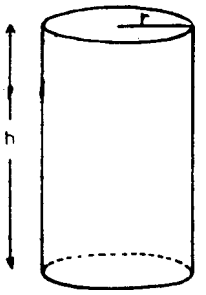

Fig. 4·9.

Since the volume is 16 π, for a given r,h is determined by,

$$\pi r^2 h = 16\pi. \qquad \therefore \ h = \frac{16}{r^2}\ .$$

$$\therefore \qquad S = 2\pi r^2 + 2\pi r.\ \frac{16}{r^2}$$

$$= 2\pi r^2 + \frac{32\pi}{r}\ .$$

Clearly, S is a function of r.

$\therefore$ S has a minimum if

$$\frac{dS}{dr} = 0 \text{ and } \frac{d^2S}{dr^2} > 0.$$

Now $\frac{dS}{dr} = 4\pi r - \frac{32\pi}{r^2}$ and $\frac{d^2S}{dr^2} = 4\pi + \frac{64\pi}{r^3}\ .$

$$\frac{dS}{dr} = 4\pi r - \frac{32\ \pi}{r^2} = 0 \text{ implies. } r^3 = 8 \text{ or } r = 2.$$

Also, since $r > 0, \left(\frac{d^2S}{dr^2}\right) = 4\pi + \frac{64\pi}{r^3} > 0.$

$\therefore$ For $r = 2$, $\frac{dS}{dr} = 0$ and $\frac{d^2S}{dr^2} > 0$,

$\therefore$ S is a minimum for $r = 2$ and $h = \frac{16}{r^2} = 4$.

$\therefore$ The cylinder of volume 16π, with minimum total surface has height 4 and has radius of the base 2.

Ex. 4. Divide the number 64 into parts such that their product is maximum.

Sol. Let the two parts of 64 be x and $64-x$ and

let $f(x) = x(64-x) = 64x - x^2$.

Clearly, $f'(x) = 64 - 2x$ and

$f''(x) = -2$.

Now, $f'(x) = 0$ implies, $64 - 2x = 0$

$\therefore$ $x = 32$.

Also, $f''(32) = -2 < 0$.

$\therefore$ $f'(32) = 0$ and $f''(32) < 0$.

$\therefore$ f has a maximum at $x = 32$.

EXERCISE 4·6

1. Investigate the following functions for maxima and minima :

(*i*) $x^3 - 2x^2 + x + 10$; (*ii*) $x^4 - 2x^2 + 1$;

(*iii*) xe^{-x} ; (*iv*) $x \log_e x$;

(*v*) $3x^4 - 2x^3 + 10$; (*vi*) $3 + 2x - x^2$;

(*v*) $2x^4 - x^2 + 1$.

2. An isosceles triangle is inscribed in a circle. Show that if the area of the triangle is a maximum, then it must be equilateral.

3. Show that the rectangle, inscribed in a circle and having a maximum perimeter, is a square.

4. A right angled triangle has hypotenuse a. Find the lengths of its sides if its area is maximum.

5. Find the length of the right cylinder with greatest lateral surface which can be inscribed in a sphere of radius R.

6. A topless box of maximum volume is to be made by cutting off equal squares at the corners of a square tin sheet of side a, and then bending the tin so as to form the sides of the box. What will be the length of the sides of the square cut off ?

7. Show that a conic tent of a given volume requires the least material when its height is $\sqrt{2}$ times the radius of its base.

8. The strength of a beam with rectangular cross section varies as the product of its breadth and the square of its depth. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius a.

9. An open cylindrical tank having a circular base and capacity, 1,000 π cub. metres is to be constructed out of sheet metal. Determine the dimensions of the tank if minimum quantity of the metal sheet is to be used.

10 The force extended by the current in a coil of radius r on a small magnet whose axis coincides with that of the coil is $\frac{kx}{(x^2+r^2)^{5/2}}$, x being the distance of the magnet from the coil. Show that the force is maximum when $x = \frac{r}{2}$. (Take $k > 0$).

11. Out of a sheet of metal in the shape of a circle of radius R, a sector is cut out and bent in the form of a cone. Show that, if the volume of the cone is maximum; then the angle of the sector is $2\pi\sqrt{2/3}$.

12. Let $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ be any real numbers. Show that the sum

$$S = \prod_{i=1}^{n} (x - \alpha_i)^2$$

is maximum if $x = \frac{\alpha_1 + \alpha_2 + \ldots + \alpha_n}{n}$.

13. A reservoir with a square bottom and open top is to be lined inside by lead. What are the dimensions of the reservoir that will require the smallest amount of lead if the capacity of the reservoir is 4 cub. meters.

ANSWERS

1. (*i*) Minimum at $x = 1$; maximum at $x = \frac{1}{3}, \frac{31}{27}$;

(*ii*) maximum at $x = 0$; minimum at $x = \pm 1, 0$.

(*iii*) maximum at $x = 1$; (*iv*) minimum at $x = \frac{1}{e}$;

(*v*) minimum at $x = \frac{1}{2}$, $\frac{159}{16}$; no decision at $x = 0$;

(*vi*) minimum at $x = 1, 4$;

(*vii*) maximum at $x = 0$; 1, minimum at $x = \pm\frac{1}{2}$; $\frac{7}{8}$.

4. $\frac{9}{\sqrt{2}}$; **5.** $R\sqrt{2}$; **8.** $\frac{2a}{\sqrt{3}}, 2a\sqrt{\frac{2}{3}}$; **9.** $r = h = 10$.

13. Side of base is 2m., height is 1 m.

4·7. Rolle's and Lagrange's Theorems

We shall now consider two important theorems known as Rolle's theorem and Lagrange's Mean Value theorem. These theorems have many applications in Analysis and in Number Theory.

Rolle's Theorem

If a function f is

(*i*) continuous on $[a, b]$,

(*ii*) differentiable on (a, b) and

(*iii*) $f(a)=f(b)$

then there exists at least one point c in (a, b) such that $f'(c)=0$.

Now, $f'(c)=0$ implies the slope of the tangent to the curve at the point $x=c$ is zero. This means the tangent to the curve at $x=c$ is parallel to the X-axis.

Therefore, if a curve $y=f(x)$ is continuous on $[a, b]$, differentiable on (a, b) and $f(a)=f(b)$, then there is at least one point on the curve at which the tangent to the curve is parallel to the X-axis.

Lagrange's Mean Value Theorem

If a function f is

(*i*) continuous on $[a, b]$ and

(*ii*) differentiable on (a, b) then there exists a point c in (a, b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Geometrical Interpretation

Let $y=f(x)$ be a function which satisfies the conditions of the Lagrange's Mean Value Theorem. This means that f is continuous on $[a, b]$ and differentiable on (a, b).

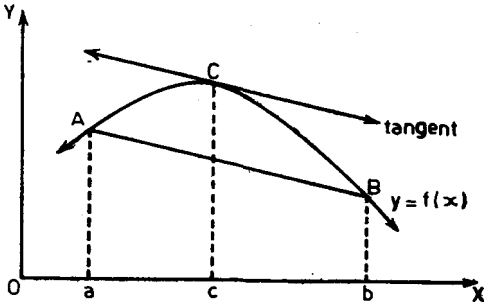

Fig. 4.10

Thus, by the theorem, there exists a c in (a, b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \qquad \ldots(1)$$

Suppose now that A and B are the end points of the curve $y=f(x)$ on the interval $[a, b]$, as shown in Fig. 4.10.

Clearly, then $A\equiv(a, f(a))$ and

$$B\equiv(b, f(b)).$$

Now, the slope of the chord AB is

$$\frac{f(b)-f(a)}{b-a} \qquad \ldots(2)$$

From (1) and (2), it follows that the chord AB of the curve $y=f(x)$ has the slope $f'(c)$. But $f'(c)$ is the slope of the tangent to the curve $y=f(x)$ at $x=c$.

Therefore, Lagranges' Mean Value theorem states that if $y=f(x)$ is continuous on $[a, b]$, differentiable on (a, b), then there exists at least one tangent to the curve which is parallel to the chord joining the end points of the curve.

Note. Let a function f satisfy the conditions of Lagranges' Mean Value Theorem on $[a, b]$. Then there exists a point c in (a, b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Now, if $f(a)=f(b)$, then clearly

$$f'(c)=0$$

which is precisely the Rolles' Theorem.

Therefore Rolles' Theorem is a special case of the Lagranges' Mean Value Theorem.

Illustrative Examples

Ex. 1. Verify Rolles' Theorem for the function

$f(x)= x^2-x-6$ on the interval $[-2, 3]$.

Sol. Since $f(x)=x^2-x-6$ is a polynomial, it is continuous and differentiable on **R**. In particular, f is continuous on $[-2, 3]$ and differentiable on $(-2, 3)$.

Also, $f(-2)=(-2)^2-(-2)-6=0.$

and $f(3)=3^3-(3)-6=0.$

$\therefore \quad f(-2)=f(3).$

$\therefore$ All the conditions of Rolles' Theorem are satisfied by the function f on $[-2, 3]$, and therefore there exists a point c in $(-2, 3)$ such that $f'(c)=0$.

Now, $f'(x)=2x-1$.

$\therefore$ $f'(c)=0$ implies $2c-1=0$ or $c=\frac{1}{2}$.

Clearly $c=\frac{1}{2}$ is a point of $(-2, 3)$.

Thus we have verified Rolle's Theorem for the function f on $[-2, 3]$.

Ex. 2. Discuss the applicability of Rolles' Theorem to the following functions.

(*i*) $f(x)=|x|$; $x \in [-1, 1]$

(*ii*) $f(x)=x^2-5x+6$; $x \in [-3, 3]$

Sol. (*i*) We know that $f(x)=|x|$ is continuous for all values of x and therefore, f is continuous on $[-1, 1]$.

But f is not differentiable at $x=0$ and $0 \in (-1, 1)$. Hence f is not differentiable on $(-1, 1)$.

It is now evident that Rolles' Theorem cannot be applied to the function $f(x)=|x|$ on $[-1, 1]$.

(*ii*) Since $f(x)=x^2-5x+6$ is a polynomial, it is continuous and differentiable on **R**. In particular, f is continuous on $[-3, 3]$ and differentiable on $(-3, 3)$.

Now, $f(-3)=(-3)^2-5(-3)+6=36$

and $f(3)=3^2-5(3)+6=0$.

$\therefore$ $f(-3) \neq f(3)$.

$\therefore$ Rolles' Theorem can not be applied to the function f over $[-3, 3]$.

Ex. 3. Verify that $f(x)=12+4x-x^2$ satisfies the conditions of Lagranges' mean value theorem on $[3, 6]$. Find the corresponding value of c.

Sol. Since $f(x)=12+4x-x^2$ is a polynomial in x, it is continuous and differentiable on **R**, and consequently it is continuous on $[3, 6]$ and differentiable on $(3, 6)$. Therefore, Lagranges' mean value to the function f is applicable on $[3, 6]$.

Now, $f'(x)=4-2x$ and therefore,

$f'(c)=4-2c$.

Also, $f(3)=12+4(3)-3^2=15$

and $f(6)=12+4(6)-6^2=0$

Putting $f'(c)=\dfrac{f(6)-f(3)}{6-3}$, we get

$$4-2c=\frac{0-15}{6-3}=-5$$

$$2c=9 \quad \text{or} \quad c=\frac{9}{2}.$$

Ex. 4. Find whether $f(x)=x^3-3x^2+6x$ satisfies the conditions of Lagranges' mean value theorem on [0, 2]. Find the value of c in case it does.

Sol. Since $f(x)=x^3-3x^2+6x$ is a polynomial function, it is continuous and differentiable on R. In particular, f is continuous on [0, 2] and differentiable on (0, 2). Clearly Lagranges' mean value theorem is applicable to the function f on [0, 2].

Now, $f'(x)=3x^2-6x+6$ and, therefore

$f'(c)=3c^2-6c+6$ for any c in (0, 2)

Also, $f(0)=0$ and

$f(2)=(2)^3-3(2)^2+6(2)=8.$

Putting now $f'(c)=\dfrac{f(2)-f(0)}{2-0}$, we get

$$3c^2-6c+6=\frac{8-0}{2-0}=4.$$

$\therefore \quad 3c^2-6c+2=0$

$$\therefore \quad c=\frac{6\pm\sqrt{36-4\times3\times2}}{2\times3}=\frac{6\pm\sqrt{36-24}}{6}$$

$$=\frac{6\pm\sqrt{12}}{6}=1\pm\frac{1}{\sqrt{3}}$$

$$\therefore \quad c=1+\frac{1}{\sqrt{3}} \quad \text{and} \quad c=1-\frac{1}{\sqrt{3}}.$$

Clearly both values of c lie in [0, 2]

There are two tangents to the curve parallel to the chord joining the end points.

Ex. 5. With the help of Lagranges' mean value theorem, show that

$$|\tan^{-1}x_1-\tan^{-1}x_2| \leqslant |x_1-x_2|, \text{ for all } x_1, x_2 \in \mathbf{R}$$

Sol. Let $f(x)=\tan^{-1}x$. Then clearly f is continuous and differentiable on R.

Now, Suppose that $x_1<x_2$. Then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Therefore, Lagranges' mean value

theorem is applicable to the function f on $[x_1, x_2]$. Hence, there exists a c in (x_1, x_2) such that

$$f'(c)=\frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

Now, $\quad f'(x)=\frac{d}{dx}\,[tan^{-1}x]=\frac{1}{1+x^2}.$

$\therefore \quad f'(c)=\frac{1}{1+c^2}.$

Then, $\quad f'(c)=\frac{f(x_2)-f(x_1)}{x_2-x_1}$ implies that

$$\frac{1}{1+c^2}=\frac{tan^{-1}x_2-tan^{-1}x_1}{x_2-x_1}$$

$\therefore \quad tan^{-1}x_2-tan^{-1}x_1=\frac{x_2-x_1}{1+c^2}$; c is a number between x_1 and x_2.

$$\therefore \quad |\,tan^{-1}x_1-tan^{-1}x_2\,| = \frac{|\,x_1-x_2\,|}{1+c^2}$$

$$\leqslant |\,x_1-x_2\,| \qquad (\because\ 1+c^2\geqslant 1)$$

$$\therefore \quad |\,tan^{-1}x_1-tan^{-1}x_2\,| \leqslant |\,x_1-x_2\,|.$$

Ex. 6. $A(0, 3)$ and $B(4, 5)$ are the two points on the locus $y=\sqrt{x^2+9}$. Prove that there is a point P on the locus between A and B such that the tangent at P is parallel to AB and find the coordinates of P.

Sol. Let $y=f(x)=\sqrt{x^2+9}$.

Clearly, the function f is continuous on $[0, 4]$ and differentiable on the interval $(0, 4)$. Therefore, Lagranges' mean value theorem is applicable to the function f on the interval $(0, 4)$.

$\therefore$ There exists a point c in $(0, 4)$ such that

$$f'(c)=\frac{f(4)-f(0)}{4-0}.$$

Now, $\quad f'(x)=\frac{x}{\sqrt{x^2+9}}$ and hence

$$f'(c)=\frac{c}{\sqrt{c^2+9}}.$$

Also, $\quad f(0)=3$ and $f(4)=5$.

$\therefore \quad \frac{f(4)-f(0)}{4-0}=f'(c)$ implies that

$$\frac{5-3}{4}=\frac{c}{\sqrt{c^2+9}}.$$

$\therefore \quad \sqrt{c^2+9}=2c.$

$\therefore \quad c^2+9=4c^2$

$\therefore \quad 3c^2=9$

$\therefore \quad c=\pm\sqrt{3}$

But only $\sqrt{3}$ is in $(0, 4)$.

$\therefore$ The tangent at the point where $x=\sqrt{3}$ is parallel to the chord AB.

But $x=\sqrt{3}$ implies $y=\sqrt{3+9}=\sqrt{12}$.

$\therefore$ The tangent at the point $(\sqrt{3}, \sqrt{12})$ is parallel to the chord AB of the curve $y=\sqrt{x^2+9}$

EXERCISE 4·7

1. Discuss the applicability of Rolle's theoren to the following functions over the correspoding intervals.

(*i*) $f(x)=x^2-7x+12$; $[-5, 7]$.

(*ii*) $f(x)=|x|.(1-x)$; $[0, 1]$.

(*iii*) $f(x)=\sqrt{25-x^2}$; $[-3, 3]$.

(*iv*) $f(x)=x^2-2x-15$; $[-3, 5]$

(*v*) $f(x)=x^3-4x^2-9x+18$; $[-2, 3]$

(*xi*) $f(x)=x^3-5x^2-x+5$; $[-1, 5]$

2. Examine whether the following functions satisfy the conditions of the lagrange's mean velue theorem on the intervals specified find the value of c, in case the theorem is applicable.

(*i*) $f(x)=x^2-4x+3$; $[1, 2]$

(*ii*) $f(x)=\frac{x+1}{x-1}$; $x\neq 1$

$=0$; $x=0$ on $[0, 2]$.

(*iii*) $f(x)=\log_e x$; $[1, e]$.

(*iv*) $f(x)=x(x-1)(x-2)$; $\left[0, \frac{1}{2}\right]$.

(*v*) $f(x)=\sqrt{25-x^2}$; $[0, 4]$.

3. A $(0, 1)$ and B $\left(\frac{\pi}{2}, 1\right)$ are the two points on the locus represented by $y=2\sin x+\cos 2x$. Prove that there is a point P on the locus between A and B such that the tangent at P is parallel to AB. Find the coordinates of P.

4 Find the coordinates of the point on the locus $y=x^2+3x+2$, the tangent at which is parallel to the chord of the locus joining the points (1, 6) and (3, 20) on it.

5. A (0, 4) and B (3, 5) are two points on the locus represented by $y=\sqrt{x^2+16}$. Prove that there is a point P on the locus between A and B such that the tangent at P is parallel to AB. Find the co-ordinates of P.

6 Prove that any chord of the curve $y=ax^2+bx+c$ is parallel to the tangent at the point whose abscissa is the same as that of the midpoint of the chord.

ANSWERS

1. (*i*) Not applicable ; (*ii*) applicable ;
(*iii*) applicable ; (*iv*) applicable ;
(*v*) applicable ; (*vi*) applicable ;

2. (*i*) applicable ; $c=\frac{3}{2}$ (*ii*) f is discontinuous *at* $x=1$, theorem not applicable ; (*iii*) applicable ; $c=e-1$; (*iv*) theorem applicable ; $c=\frac{\sqrt{12}-\sqrt{7}}{\sqrt{12}}$; (*v*) applicable, $c=\sqrt{5}$; 3. $\left(\frac{\pi}{6}, -\frac{3}{2}\right)$ 4. (2, 12), 5 $(\sqrt{2}, 3\sqrt{2})$,

5

Indefinite Integrals

5·1. Preliminaries

It was discussed in the XIth standard that if f and g are two functions such that $f(x)=g'(x)$, then g is called *integral* of the function f, and this is written as

$$\int f(x)dx=g(x).$$

We have also seen that

$$\int f(x)dx=g(x)+c,$$

where c is any arbitrary constant and is called *the constant of integration.*

Illustrative Examples

Ex. 1. We know that

$$\frac{d}{dx}\left[\tan x\right]=\sec^2 x.$$

$$\therefore \quad \int \sec^2 x\, dx=\tan x+c\,;$$

c is the constant of integration.

Ex. 2. We have,

$$\frac{d}{dx}\left[\tfrac{1}{2}\log_e (3x^2+4x-5)+c\right]=\frac{3x+2}{3x^2+4x5}.$$

$$\therefore \quad \int \frac{3x+2}{3x^2+4x-5}\,dx=\tfrac{1}{2}\log\,(3x^2+4x-5)+c.$$

EXERCISE 5·1.

Using the above definition of integral, show that :

(*i*) $\int (x^3+5x^2+2x^{-2})\,dx=\frac{1}{12}\left(3x^4+20x^3-\frac{24}{x}\right)+c,$

(*ii*) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+c.$

(*iii*) $\int \frac{1}{1+\sin x}\,dx=\tan x-\sec x+c.$

(*iv*) $\int x\cos x\,dx=x\sin x+\cos x+c.$

(*v*) $\int \log_e x\,dx=x(\log x-1)+c.$

(*vi*) $\int \cot x\,dx=\log(\sin x)+c.$

(*vii*) $\int e^x\,dx=e^x+c.$

(*viii*) $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx=\frac{(\sin^{-1}x)^2}{2}+c.$

It is quite clear that we can use the basic definition of integral to write the integral of *some known* functions. However if we want to write the integral of an arbitrary function, the knowledge of only the definition of integral will not be sufficient. It is for th s reason, we shall develop the techniques of *integration* (the process of obtaining the integrals of given functions) in this chapter. We shall begin with *some rules* and *standard formulae*.

5·2. Some Rules of Integration and Some Standard Formulae

The following rules are found to be useful in finding the integrals of functions.

Rule I. $\int\left[f_1(x)+f_2(x)\right]dx=\int f_1(x)\,dx+\int f_2(x)\,dx$

Proof. Let $\int f_1(x)\,dx=g_1(x)$

and $\int f_2(x)\,dx=g_2(x)$

$\therefore$ By definition of integrals, we have

$$\frac{d}{dx}\left[g_1(x)\right]=f_1(x)$$

and $$\frac{d}{dx}\left[g_2(x)\right]=f_2(x)$$

But, we know that

$$\frac{d}{dx}\Big[g_1(x)+g_2(x)\Big]=\frac{d}{dx}[g_1(x)]+\frac{d}{dx}[g_2(x)]$$

$\therefore$ We have

$$\frac{d}{dx}[g_1(x)+g_2(x)]=[f_1(x)+f_2(x)].$$

Applying once again the definition of the integrals, we get

$$\int[f_1(x)+f_2(x)]\,dx=g_1(x)+g_2(x)$$

$$=\int f_1(x)\,dx+\int f_2(x)\,dx.$$

$$\therefore \quad \int[f_1(x)+f_2(x)]\,dx=\int f_1(x)\,dx+\int f_2(x)\,dx.$$

Rule II. $\int k\,f(x)\,dx=k\int f(x)\,dx.$

Proof. Let $\int f(x)\,dx=g(x).$

Then, clearly

$$\frac{d}{dx}[g(x)]=f(x)$$

Now, we know that

$$\frac{d}{dx}\,[k\,g(x)]=k\frac{d}{dx}[g(x)]$$

$$\therefore \quad \frac{dx}{dx}\,[k\,g(x)]=k\,f(x)$$

$\therefore$ Using the definition of integral, we get

$$\int k\,f(x)\,dx=k\,g(x)$$

$$=k\int f(x)\,dx.$$

$$\therefore \quad \int k\,f(x)\,dx=k\int f(x)\,dx.$$

We shall now consider some *standard formulae* which we shall use from time to time in the discussion of the techniques of integration. We list them below.

Some Standard Formulae

1. $\int x^n\,dx=\frac{x^{n+1}}{n+1}+c\,; \qquad n\neq -1.$

2. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c\,; \qquad n \neq -1.$

3. $\int \frac{1}{x}\,dx = \log_e |\,x\,| + c$

4. $\int \frac{1}{ax+b}\,dx = \frac{\log_e |\,ax+b\,|}{a} + c.$

5. $\int \cos x\,dx = \sin x + c.$

6. $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c.$

7. $\int \sin x\,dx = -\cos x + c.$

8. $\int \sin(ax+b)\,dx = \frac{-\cos(ax+b)}{a} + c.$

9. $\int \sec^2 x\,dx = \tan x + c.$

10. $\int \sec^2(ax+b)\,dx = \frac{\tan(ax+b)}{a} + c.$

11. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c.$

12. $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{\cot(ax+b)}{a} + c.$

13. $\int \sec x \tan x\,dx = \sec x + c$

14. $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{\sec(ax+b)}{a} + c.$

15. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c.$

16. $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = \frac{-\operatorname{cosec}(ax+b)}{b} + c.$

17. $\int e^x\,dx = e^x + c.$

18. $\int e^{(ax+b)}\,dx = \frac{1}{a}\,e^{(ax+b)} + c.$

19. $\int a^x\,dx = \frac{a^x}{\log_e a} + c.$

20. $\int a^{(bx+k)} = \frac{a^{(bx+k)}}{b \log_e a} + c.$

21. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c.$

22, $\int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1} x + c.$

23. $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c.$

24. $\int \frac{1}{1+x^2}\,dx = -\cot^{-1} x + c.$

25. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + c.$

26. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = -\operatorname{cosec}^{-1} x + c.$

We use the above formulae in the following examples.

Illustrative Examples

Ex. 1. Evaluate integrals of the following functions :

(*i*) $(3-4x)^{1/2}$; (*ii*) $\frac{5}{(2-3x)}$; (*iii*) $3^x 2^x$

Sol.

(*i*) $\int (3-4x)^{1/2}\,dx = \frac{(3-4x)^{(\frac{1}{2}+1)}}{(-4)(\frac{1}{2}+1)} + c.$

$= \frac{(3-4x)^{3/2}}{(-4\times 3/2)} + c.$

$= \frac{-(3-4x)^{3/2}}{3} + c.$

(*ii*) $\int \frac{5}{(2-3x)}\,dx = 5\int \frac{1}{(2-3x)}\,dx$ (By Rule II)

$= \frac{5\log_e |2-3x|}{-3} + c.$

$= -\frac{5}{3}\log_e |2-3x| + c.$

(*iii*) $\int 3^x 2^x dx = \int (32)^x\,dx$

$= \frac{(32)^x}{\log_e 32} + c.$

Ex. 2. Obtain the integrals of the following functions :

(*i*) $e^{(5x+6)}$; (*ii*) $\sec^2(4-3x)$;

(*iii*) $\sec^2(2x+3)+\cos(4-5x)$;

(*iv*) $\frac{5}{\sqrt{1-x^2}} + 3\sin x$

Sol. (*i*) $\int e^{(5x+6)}\,dx = \frac{e^{(5x+6)}}{5} + c.$

(*ii*) $\int \sec^2(4-3x)\,dx = \frac{\tan(4-3x)}{-3} + c.$

$= -\frac{1}{3}\tan(4-3x) + c.$

(*iii*) $\int [\sec^2(2x+3) + \cos(4-5x)]\,dx$

$= \int \sec^2(2x+3)\,dx + \int \cos(4-5x)\,dx$ by Rule I

$= \frac{\tan(2x+3)}{2} + \frac{\sin(4-5x)}{-5} + c.$

$= \frac{1}{2}\tan(2x+3) - \frac{1}{5}\sin(4-5x) + c.$

(*iv*) $\int \left[\frac{5}{\sqrt{1-x^2}} + 3\sin x\right] dx$

$= \int \frac{5}{\sqrt{1-x^2}}\,dx + \int 3\sin x\,dx$ **(by Rule I)**

$= 5\int \frac{1}{\sqrt{1-x^2}}\,dx + 3\int \sin x\,dx$

(by Rule II)

$= 5\sin^{-1}x + 3(-\cos x) + c.$

$= 5\sin^{-1}x - 3\cos x + c.$

EXERCISE 5·2.

Find the integrals of the following functions ;

1. (*i*) $(5x+2)^{-2}$; (*ii*) $(7-3x)^{3/2}$;
   (*iii*) $(2-x)^{-5/4}$
2. (*i*) $\frac{1}{4x-3}$; (*ii*) $\frac{4}{3-2x}$
   (*iii*) $\frac{7}{2-x}$.
3. (*i*) $\sin(4x+3)$; (*ii*) $\cos(5-3x)$;
   (*iii*) $\sec^2(4-x)$; (*iv*) $\sec(2x+3)\tan(2x+3)$;
   (*v*) $\operatorname{cosec}^2(3-5x)$;
4. (*i*) 2^x ; (*ii*) $3^x\,2^x$;
   (*iii*) $4^x e^x$; (*iv*) a^{2x} ;
   (*v*) e^{mx} ;
5. (*i*) $a^{2x} + \sec^2 x$; (*ii*) $\frac{1}{1+x^2} + \frac{\sin x}{\cos^2 x}$;
   (*iii*) $\frac{1}{\sqrt{1-x^2}} - \frac{\cos x}{\sin x^2}$

6. (*i*) $e^{2x}-\frac{1}{x\sqrt{x^2-1}}$; (*ii*) $(2-3x)^{-5/2}-\text{cosec}^2(2x-1)$;

(*iii*) $\sqrt{3-2x}+3\cos x-\frac{4}{\sqrt{1-x^2}}$.

7. (*i*) $3^x-4\sec(x-2)\tan(x-2)$;

(*ii*) $4e^{4x}+3\cos(3x-4)-4\sin(3-4x)+\frac{7}{\cos^2(5x-2)}$.

ANSWERS

1. (*i*) $\frac{(5x+2)^{-1}}{5}+c$. (*ii*) $\frac{2}{15}(7-3x)^{5/2}+c$.,

(*iii*) $4(2-x)^{1/4}+c$. ;

2. (*i*) $-\frac{1}{4}\log_e|4x-3|+c$. ; (*ii*) $2\log_e|3-2x|+c$. ;

(*iii*) $-7\log_e|2-x|+c$.

3. (*i*) $-\frac{1}{4}\cos(4x+3)+c$. ; (*ii*) $-\frac{1}{3}\sin(5-3x)+c$. ;

(*iii*) $-\tan(4-x)+c$. ; (*iv*) $\frac{1}{2}\sec(2x+3)+c$. ;

(*v*) $\frac{1}{5}\cot(3-5x)+c$.

4. (*i*) $\frac{1}{\log_e 2}(2^x)+c$. ; (*ii*) $\frac{3^x 2^x}{\log_e 6}+c$. ;

(*iii*) $\frac{4^x e^x}{1+\log_e 4}+c$. ; (*iv*) $\frac{a^{2x}}{\log_e a}+c$. ;

(*v*) $\frac{1}{m}e^{mx}+c$. ;

5. (*i*) $\frac{a^{3x}}{3\log_e a}+\tan x+c$. (*ii*) $\tan^{-1}x+\sec x+c$.

(*iii*) $\sin^{-1}x+\cot x+c$.

6. (*i*) $\frac{e^{2x}}{2}-\text{cosec}^{-1}x+c$. ; (*ii*) $\frac{2}{9}(2-3x)^{-3/2}+\frac{1}{2}\cot(2x-1)+c$;

(*iii*) $-\frac{1}{3}(3-2x)^{3/2}+3\sin x-4\sin^{-1}x+c$.

7. (*i*) $\frac{3^x}{\log_e 3}-4\sec(x-2)+c$;

(*ii*) $e^{4x}+\sin(3x-4)+\cos(3-4x)+\frac{7}{5}\tan(5x-2)+c$;

5·3. Integrals of Some Algebraic and Trigonometric functions.

1. Algebraic functions.

Algebraic functions of the type $\frac{f(x)}{px+q}$ where $f(x)$ is a polynomial in x can be expressed as a combination of functions in the standard form and then can be integrated very easily.

Ex. 1. Integrate the following functions :

(*i*) $\frac{2x+5}{2x-1}$; (*ii*) $\frac{3x+4}{2x+1}$;

(*iii*) $\frac{5x^2-4x+3}{2x+3}$.

Sol. (*i*) $\int\frac{2x+5}{2x-1}\,dx=\int\frac{(2x-1)+6}{2x-1}\,dx.$

$$=\int\left[1+\frac{6}{2x-1}\right]dx.$$

$$=\int 1dx+6\int\frac{1}{2x-1}\,dx.$$

$$=x+\frac{6\log_e|2x-1|}{2}+c.$$

$$=x+3\log_e|2x-1|+c.$$

(*ii*) $\int\frac{3x+4}{2x+1}\,dx=\int\frac{\frac{3}{2}(2x+\frac{8}{3})}{2x+1}\,dx.$

$$=\int\frac{\frac{3}{2}(2x+1+\frac{5}{3})}{(2x+1)}\,dx.$$

$$=\tfrac{3}{2}\int 1.\,dx+\tfrac{5}{3}\int\frac{1}{2x+1}\,dx.$$

$$=\tfrac{3}{2}x+\tfrac{5}{3}\left[\frac{\log_e|2x+1|}{2}\right]+c.$$

$$=\tfrac{3}{2}x+\tfrac{5}{6}\log_e|2x+1|+c.$$

(*iii*) Here, we divide the numerator by the denominator.

$$2x+3)\ 5x^2-4x+3\ \left(\frac{5}{2}x-\frac{23}{4}.\right.$$

$$\underline{-\ 5x^2+\frac{15}{2}x.}$$

$$-\frac{23}{2}x+3$$

$$\underline{-\frac{23}{2}x-\frac{69}{4}}$$

$$\frac{81}{4}$$

$$\therefore\quad\int\frac{5x^2-4x+3}{2x+3}\,dx=\int\frac{\left(\frac{5}{2}x-\frac{23}{4}\right)(2x+3)+\frac{81}{4}}{(2x+3)}\,dx$$

$$= \frac{5}{2}x - \frac{23}{4} + \frac{(81/4)}{(2x+3)}\Big] dx.$$

$$= \frac{5x^2}{4} - \frac{23}{4}x + \frac{81}{4}\left|\frac{\log_e |2x+3|}{2}\right| + c$$

$$= \frac{5x^2}{4} - \frac{23}{4}x + \frac{81}{8}\log_e |2x+3| + c.$$

EXERCISE 5·3. (*a*)

Integrate the following functions with respect to x ;

1. (*i*) $\frac{x+2}{x+1}$; (*ii*) $\frac{2x+3}{x-1}$;

(*iii*) $\frac{6x-5}{2x-3}$; (*iv*) $\frac{7x+5}{3x-2}$;

(*vi*) $\frac{2x+2}{3x-1}$;

2. (*i*) $\frac{3x^2+5x-2}{x-1}$; (*ii*) $\frac{4x^2-7x+5}{2x+1}$;

(*iii*) $\frac{3x^2+6x-2}{2x+1}$; (*iv*) $\frac{7x^2-8x+3}{3x-1}$;

(*v*) $\frac{x^3+4x^2-7x+5}{x+2}$;

ANSWERS

1. (*i*) $x+\log_e |x+1| + c$; (*ii*) $2x+5\log_e |x-1| + c$;

(*iii*) $3x-\frac{1}{2}\log_e |2x-1| + c$; (*iv*) $\frac{7}{3}x+\frac{29}{9}\log_e |3x-2| + c$.

(*v*) $\frac{2}{3}x+\frac{11}{9}\log_e |3x-1| + c$.

2. (*i*) $\frac{3x^2}{2}+8x+6\log_e |x-1| + c$;

(*ii*) $2x^2-\frac{9}{2}x+\frac{1}{4}\log_e |2x+1| + c$;

(*iii*) $\frac{3}{4}x^2+\frac{21}{4}x+\frac{55}{8}\log_e |2x-3| + c$;

(*iv*) $\frac{7}{6}x^2-\frac{17}{9}x+\frac{10}{27}\log_e |3x-1| + c$;

(*v*) $\frac{x^3}{3}+x^2-11x+27\log_e |x+2| + c$:

2. Trigonometric functions.

We know that $\sin x$, $\cos x$, $\sin(ax+b)$ and $\cos(ax+b)$ can be integrated using standard formulae. But $\sin^2 x$, $\cos^2 x$, $\sin^2(ax+b)$, $\cos^2(ax+b)$, and other higher integral powers of sine and cosine functions can not be integrated directly. In such cases, we make use of trigonometric identities and other trigonometric results in bringing these expressions to the desired form.

for example, $\cos^2 x$ can be expressed as $\left(\frac{1+\cos 2x}{2}\right)$ and the function in the later form can be integrated. We list below some of the trigonometric identities which are useful.

1. $\sin 2x = 2 \sin x \cos x$;
2. $\sin^2 x = \frac{1-\cos 2x}{2}$;
3. $\cos^2 x = \frac{1+\cos 2x}{2}$;
4. $\sin A \cos B = \frac{1}{2}\left[\sin (A+B) + \sin (A-B)\right]$;
5. $\cos A \cos B = \frac{1}{2}\left[\cos (A+B) - \cos (A+B)\right]$;
6. $\sin A \sin B = \frac{1}{2}\left[\cos (A-B) - \cos (A+B)\right]$;
7. $1+\sin x = \left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2$;
8. $1-\sin x = \left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)^2$;
9. $\tan^2 x = \sec^2 x - 1$;
10. $\cot^2 x = \operatorname{cosec}^2 x - 1$

Illustrative Examples.

Ex. 1. $\int \tan^2 x \, dx = \int (\sec^2 x - 1)\, dx$

$= \int \sec^2 x \, dx - \int 1 \, dx$

$= \tan x - x + c.$

Ex. 2. $\int \cos^2 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right) dx.$

$= \frac{1}{2}\int 1 . \, dx + \frac{1}{2}\int \cos 2x \, dx.$

$= \frac{1}{2} x + \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + c.$

$= \frac{1}{2} x + \frac{1}{4} \sin 2x + c.$

Ex. 3. $\int \sqrt{1-\sin x}\, dx = \int \left(\cos \frac{x}{2} - \sin \frac{x}{2}\right) dx.$

$= 2\left[\sin \left(\frac{x}{2}\right) + \cos \left(\frac{x}{2}\right)\right] + c.$

Ex. 4. $\int \frac{\cos x}{1+\cos x} \cdot dx = \int \frac{\cos x\,(1-\cos x)}{(1+\cos x)\,(1-\cos x)}\,dx.$

$$= \int \frac{\cos x - \cos^2 x}{1-\cos^2 x}\,dx.$$

$$= \int \frac{\cos x - \cos^2 x}{\sin^2 x}\,dx.$$

$$= \int \left[\frac{\cos x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}\right] dx.$$

$$= \int [\operatorname{cosec} x \cot x - \cot^2 x]\,dx.$$

$$= \int \operatorname{cosec} x \cot x\,dx - \int \cot^2 x\,dx$$

$$= \int \operatorname{cosec} x \cot x\,dx - \int (\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\operatorname{cosec} x + \cot x + x + c.$$

EXERCISE 5·3 (b)

Find the integrals of the following :

1. $\cot^2 x$;
2. $\sin^2 x$;
3. $\cos^2 x - \sin^2 4x$;
4. $\sin 7x \cdot \cos x$;
5. $\cos 8x \sin 4x$;
6. $\sin 5x \cos 7x$;
7. $2 \sin 4x \sin 6x$;
8. $\cos x \cos 4x \cos 7x$;
9. $\sin^2 x \cos^2 x$;
10. $\cos^3 x$;
11. $\cos^3 2x - \sin^2 x$;
12. $\sqrt{1-\cos x}$;
13. $\sqrt{1+\cos x}$;
14. $\sqrt{1+\sin 4x}$;
15. $\frac{1}{1+\cos x}$;
16. $\frac{1}{1-\cos x}$;
17. $(\tan x + \cot x)^2$;
18. $\sqrt{1-\sin x}$;
19. $(\sin x + \cos x)^2$;
20. $\frac{\sin x}{1+\sin x}$;
21. $\frac{\tan x}{\sec x + \tan x}$;
22. $\cos^4 x$;
23. $\sin^4 x$;
24. $\sin^2 (3x+4)$;

ANSWERS

1. $-\cot x - x + c$;
2. $\frac{x}{2} - \frac{\sin 2x}{4} + c$;
3. $\frac{\sin 2x}{2} + \frac{\sin 8x}{16} + c$;
4. $\frac{-\cos 8x}{16} - \frac{\cos 6x}{12} + c$;
5. $\frac{-\cos 12x}{24} + \frac{\cos 4x}{8} + c$;
6. $\frac{\sin 12x}{24} + \frac{\sin 2x}{4} + c$.
7. $\frac{\sin 2x}{2} - \frac{\sin 10x}{10} + c$;
8. $\frac{\sin 14x}{56} + \frac{\sin 8x}{32} + \frac{\sin 6x}{48} + \frac{x}{8} + c$;
9. $\frac{x}{8} - \frac{\sin 4x}{32} + c$;

10. $\frac{3\sin x}{4}+\frac{\sin 3x}{12}+c.$ **11.** $\frac{5\sin 2x}{8}+\frac{\sin 6x}{24}-\frac{x}{2}+c$;

12. $-2\sqrt{2}\cos\frac{x}{2}+c$: **13.** $2\sqrt{2}\sin\frac{x}{2}+c$;

14. $\frac{\sin 2x}{2}-\frac{\cos 2x}{2}+c$; **15.** $\tan\frac{x}{2}+c$;

16. $-\cot\frac{x}{2}+c$; **17.** $-2\cot 2x+c$;

18. $2\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)+c$; **19.** $x-\frac{\cos 2x}{2}+c.$

20. $\sec x-\tan x+x+c$: **21.** $\sec x-\tan x+c$;

22. $\frac{3x}{8}+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+c$; **23.** $\frac{3x}{8}-\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+c$:

24. $2\tan\frac{x}{2}-x+c$; **25.** $\frac{x}{2}-\frac{(\sin 6x+8)}{12}+c$;

5·4. Methods of Integration.

The function to be integrated is called the *integrand.*

The standard formulae listed earlier are useful only in the case when the integrand are of the form given in the list or any *linear combination* of them. But the problem arises when the function to be integrated (*i.e.* the integrand) is not in that form. However, many functions can be *expressed* as a linear combinations of the functions in the standard form after some *adjustments* and *skillful maniputation* of the terms involved in the integrand. The *techniques* used to achieve this are called the *methods of integration.*

We broadly classify these methods in the following three classes.

(*i*) Integration by substitution.

(*ii*) Integration by parts.

(*iii*) Integration by partial fractions.

We shall begin with integration by substitution.

5·5. Integration by Substitution.

Consider the expression $\int f(x)\,dx$. Here we want to integrate the function f with respect to the (independent) variable x. In many cases though the function f is not in the standard form, it can be converted to a function in the standard function simply by changing the independent variable x by a *suitable substitution* $x=\phi(t)$.

We shall first prove the result concerning the *integration by substitution* in the following theorem.

Thm, If $x=\phi(t)$ is a differentiable function of t, then,

$\int f(x)\,dx=\int f[\phi(t)].\ \phi'(t)\,dt.$

Proof. Let $F(x)=\int f(x)\,dx.$ ...(1)

$\therefore$ By the definition of integral, we have

$$\frac{d}{dx}[F(x)]=f(x).$$

Now, since x is a function of t, we can use the chain rule of derivatives, which gives us

$$\frac{d}{dt}[F(x)]=\frac{d}{dx}[F(x)]\frac{dx}{dt}$$

$$=f(x)\frac{dx}{dt}.$$

$\therefore$ By definition of integral, we get.

$$F(x)=\int f(x)\frac{dx}{dt}\,dt. \quad ...(2)$$

$\therefore$ From (1) and (2), we get

$$\int f(x)\,dx=\int f(x)\frac{dx}{dt}\,dt. \quad ...(3)$$

Since $x=\phi(t)$, $\frac{dx}{dt}=\phi'(t)$.

$\therefore$ (3) becomes

$\int f(x)\,dx=\int f[\phi(t)]\,\phi'(t)\,dt.$

Note. Comparing $\int f(x)\,dx$ with $\int f[\phi(t)]\,\phi'(t)\,dt$, we observe that in the second integral x is replaced by $\phi(t)$ and dx is replaced by $\phi'(t)\,dt$. Therefore, while using the method of substitution, we follow these steps mechanically and proceed to compute the integral.

Now, we shall consider different situations in which the method of substitution can be applied to compute the integrals.

I. Integral of $f(ax+b)$ where $f(x)$ is in the standard form.

Suppose that

$\int f(x)\,dx=g(x)$, and suppose further that we have to compute the integral.

$I=\int f(ax+b)\,dx.$

In I, put $ax+b=t$.

$\therefore$ $a\,dx=dt$ or $dx=\frac{1}{a}.\,dt.$

$$\therefore \quad I=\int f(t)\,\frac{1}{a}\,dt=\frac{1}{a}\int f(t)\,dt.$$

$$=\frac{1}{a}\,g(t)+c.$$

$$=\frac{1}{a}\,g(ax+b)+c.$$

Illustrative Examples.

Ex. 1. Let $I=\int (ax+b)^n\,dx$; $n\neq -1$.

Put $t=ax+b$.

$$\therefore \quad dt=a\,dx \text{ or } dx=\frac{1}{a}\,\mathrm{dt}.$$

$$\therefore \quad I=\int\frac{1}{a}\,t^n\,dt=\frac{1}{a}\int t^n\,dt=\frac{1}{a}\left(\frac{t^{n+1}}{n+1}\right)+c.$$

$$=\frac{(ax+b)^{n+1}}{(n+1)}+c.$$

Ex. 2. Let $I=\int cos\,(ax+b)\,dx$.

Put $ax+b=t$.

$$\therefore \quad adx=dt \quad \text{or} \quad dx=\frac{1}{a}\,dt.$$

$$\therefore \quad I=\frac{1}{a}\int cos\,(t)\,dt=\frac{1}{a}\,sin\,t+c.$$

$$=\frac{sin\,(ax+b)}{a}+c.$$

Ex. 3. Let $I=\int e^{(ax+b)}\,dx.$

Put $ax+b=t$.

$$\therefore \quad adx=dt \quad \text{or} \quad dx=\frac{1}{a}\,dx.$$

$$\therefore \quad I=a\int e^t\,dt=\frac{1}{a}\,e^t+c$$

$$=\frac{1}{a}\,e^{(ax+b)}+c.$$

EXERCISE 5·5. (a)

1. $\int(5-2x)^{-3}\,dx$;
2. $\int \sin(3x-4)\,dx$;
3. $\int \cos(5-3x)\,dx$;
4. $\int \mathrm{cosec}^2(7-4x)\,dx$;
5. $\int \mathrm{cosec}\,(4x-5)\cot(4x-5)\,dx$;
6. $\int e^{4x-3}dx$;
7. $\int a^{3-2x}\,dx$;
8. $\int \frac{dx}{\sqrt{1-(3x+2)^2}}$

9. $\int \frac{dx}{1+(4x-3)^2}$ 10. $\int \frac{dx}{\sqrt{1-(4-5x)^2}}$

ANSWERS

1. $\frac{(5-2x)^{-2}}{4}+c$; 2. $\frac{-\cos(8x-4)}{8}+c$;

3. $\frac{\sin(5-3x)}{-3}+c$; 4. $\frac{\cot(7-4x)}{-4}+c$;

5. $\frac{-\text{cosec}(4x-5)}{4}+c$; 6. $\frac{e^{4x-3}}{4}+c$;

7. $\frac{a^{3-2x}}{-2\log_e a}+c$; 8. $\frac{1}{3}\sin^{-1}(3x+2)+c$.

9. $\frac{1}{4}\tan^{-1}(4x-3)+c$; 10. $-\frac{1}{5}\sin^{-1}(4-5x)+c$.

II. Integrals of the type $\int [f(x)]^n f'(x)dx$; $n \neq -1$.

Let $I=\int [f(x)]^n f'(x)\,dx$.

Put $f(x)=t$

$\therefore$ $f'(x)dx=dt$

$\therefore$ $I=\int t^n\,dt=\frac{t^{n+1}}{n+1}+c$

$=\frac{[f(x)]^{n+1}}{n+1}+c$

Illustrative Examples

Integrate the following

(*i*) $\int x^2(x^3+1)^{5/2}\,dx$; (*ii*) $\int \frac{\cos}{\sqrt{\sin x}}dx$;

(*iii*) $\int \frac{\sin 2x}{(a\cos^2 x+b\sin^2 x)^3}\,dx$; (*iv*) $\int \frac{(\sin^{-1}x)^4}{\sqrt{1-x^2}}\,dx$;

(*v*) $\int \frac{e^{k\tan^{-1}x}}{1+x^2}\,dx$

Sol. Observe that in all the above examples, the integrand is a product of two functions $[f(x)]^n$ and $f'(x)$.

(*i*) Let $I=\int x^2(x^3+1)^{5/2}\,dx=\int (x^3+1).x^2\,dx$.

Put $x^3+1=t$

$\therefore$ $3x^2\,dx=dt$ or $x^2\,dx=\frac{1}{3}\,dt$.

$\therefore$ $I=\int t^{5/2}\frac{1}{3}\,dt=\frac{1}{3}\int t^{5/2}\,dt$

$$= \frac{1}{3}\left(\frac{t^{7/2}}{\frac{7}{2}}\right)+c$$

$$= \frac{2}{21}(x^3+1)^{7/2}+c$$

(*ii*) Let $I=\int \frac{\cos x}{\sqrt{\sin x}}\,dx=\int (\sin x)^{1/2}\cos x\,dx$

Put $\sin x=t$

$\therefore \cos x\,dx=dt$

$$\therefore \quad I=\int t^{-1/2}\,dt=\frac{t^{1/2}}{\frac{1}{2}}+c$$

$$=2\sqrt{\sin x}+c$$

(*ii*) $I=\int \frac{\sin 2x}{(a\cos^2 x+b\sin^2 x)^3}\,dx$

$$=\int \frac{1}{(a\cos^2 x+b\sin^2 x)^3}\sin 2x\,dx.$$

Put $(a\cos^2 x+b\sin^2 x)=t.$

$\therefore \quad (-2a\cos x\sin x+2b\sin x\cos x)\,dx=dt.$

$\therefore \quad 2(b-a)\sin x\cos x\,dx=dt.$

$\therefore \quad \sin 2x=dx\,\frac{1}{(b-a)}\,dt.$

$$\therefore \quad I=\int \frac{1}{t^3}\cdot\frac{1}{(b-a)}\,dt=\frac{1}{(b-a)}\int t^{-3}\,dt$$

$$=\frac{1}{(b-a)}\left(\frac{t^{-2}}{-2}\right)+c$$

$$=\frac{(a\cos^2 x+b\sin^2 x)^{-2}}{2(a-b)}+c.$$

(*iv*) $I=\int \frac{(\sin^{-1}x)^4}{\sqrt{1-x^2}}\,dx=\int (\sin^{-1}x)^4\cdot\frac{1}{\sqrt{1-x^2}}\,dx.$

Put $\sin^{-1}x=t.$

$$\therefore \quad \frac{1}{\sqrt{1-x^2}}\,dx=dt.$$

$$\therefore \quad I=\int t^4\,dt=\frac{t^5}{5}+c$$

$$=\frac{(\sin^{-1}x)^5}{5}+c.$$

(v) Let $I=\int \frac{e^{k\tan^{-1}x}}{1+x^2}\,dx=\int e^{k\tan^{-1}x}\cdot\frac{1}{1+x^2}\,dx.$

Put $k\tan^{-1}=t.$

$\therefore \quad \frac{k}{1+x^2}\,dx=dt.$

$\therefore \quad \frac{1}{1+x^2}\,dx=\frac{1}{k}\,dt.$

$\therefore \quad I=\int e^t\,\frac{1}{k}\,dt=\frac{1}{k}\int e^t\,dt$

$=\frac{1}{k}\,e^t+c$

$=\frac{1}{k}\,e^{k\tan^{-1}x}+c.$

Note. Whenever we write $\log x$, we shall always mean $\log_e x$. This method we shall adopt to write the expression in a compact form.

EXERCISE 5·5 (b)

Evaluate the following integrals.

1. $2x^3\,(x^4+2)^{5/2}$;
2. $7x\sqrt{x^2-1}$;
3. $\frac{2x+3}{\sqrt{x^2+3x}}$;
4. $x^5\sqrt{a^3+x^3}$;
5. $-x^3\cos x^4$;
6. $\frac{x}{\sqrt{a^2-x^2}}$;
7. xe^{x^2} ;
8. $\frac{\sin\sqrt{x}}{\sqrt{x}}$;
9. $\frac{\sin x}{\cos^7 x}$;
10. $\sin^4 x\cos x$;
11. $\cos^8 x\,.\sin x$;
12. $\sin^4 x-\cos^3 x$;
13. $\left(\frac{\sec x}{1+\tan x}\right)^2$;
14. $\frac{e^{2x}}{\sqrt{e^{2x}+1}}$
15. $\tan^5 x.\sec^2 x$;
16. $\sec^4 x.\tan x$;
17. $\frac{\sin(\log x)}{x}$;
18. $\frac{\operatorname{cosec}^5 x}{\sec x}$;
19. $\frac{1}{x\cos^2(\log x)}$;
20. $\frac{(\tan^{-1}x)^3}{1+x^2}$;
21. $\left(1-\frac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}$
22. $\frac{e^{3n}\sec^{-1}x}{x\sqrt{(x^2-1)}}$
23. $\frac{ae^x-be^{-x}}{(ae^x+be^{-x})^4}$;
24. $\frac{[\log(\sin e^x)]\,e^x}{\tan e^x}$;
25. $\frac{2+x^2}{1+x^2}\,a^{(x+\tan^{-1}x)}$

ANSWERS

1. $\frac{(x^4+2)^{7/2}}{7}+c$; 2. $\frac{7}{3}(x^2-1)^{3/2}+c$;

3. $\sqrt{x^2+3x}+c$,

III. Integrals of f(x) $(ax+b)^n$ where f(x) is a polynomial in x.

In such cases, we put $ax+b=t$ and express $f(x)$ in terms of t. The following examples will make the procedure clear.

Illustrative Examples

Ex. 1. Obtain the integrals of the following ;

(*i*) $(7x-2)(3-4x)^{5/2}$; (*ii*) $\frac{3x^2+5x-4}{(2x-3)^7}$;

(*iii*) $\frac{(4\sin x-5)\cos x}{(7\sin x+2)^8}$

Sol. (*i*) Let $3-4x=t$.

$\therefore \quad -4dx=dt \quad$ or $\quad dx=-\frac{1}{4}dt$

Also, $x=\frac{3-t}{4}$.

$\therefore \quad 7x-2=7\left(\frac{3-t}{4}\right)-2=\frac{-7t}{4}+\frac{13}{4}$

$\therefore \quad \int(7x-2)(3-4x)^{5/2}\,dx$

$=\int\left(-\frac{7t}{4}+\frac{13}{4}\right)t^{5/2}\left(-\frac{1}{4}\right)dt$

$=\frac{7}{16}\int t^{7/2}\,dt-\frac{13}{16}\int t^{5/2}\,dt$

$=\frac{7}{16}\left(\frac{t^{9/2}}{\frac{9}{2}}\right)-\frac{13}{16}\left(\frac{t^{7/2}}{\frac{7}{2}}\right)+c$

$=\frac{7}{72}(3-4x)^{9/2}-\frac{13}{56}(3-4x)^{7/2}+c$

(*ii*) Put $2x-3=t$.

$\therefore \quad 2dx=dt \quad$ or $\quad dx=\frac{1}{2}dt$

Also, $x=\frac{t+3}{2}$.

$\therefore \quad 3x^2+5x-4=3\left(\frac{t+3}{2}\right)^2+5\left(\frac{t+3}{2}\right)-4$

$=\frac{3t^2}{4}+7t+\frac{31}{4}$

$$\therefore \int \frac{3x^2+5x-4}{(2x-3)^7}\,dx = \int\left(\frac{3t^2}{4}+7t+\frac{31}{4}\right)t^{-7}\left(\frac{1}{2}\right)dt$$

$$= \frac{3}{8}\int t^{-5}\,dt + \frac{7}{2}\int t^{-6}\,dt + \frac{41}{8}\int t^{-7}dt$$

$$= \frac{3}{8}\left(\frac{t^{-4}}{-4}\right)+\frac{7}{2}\left(\frac{t^{-5}}{-5}\right)+\frac{41}{8}\left(\frac{t^{-6}}{-6}\right)+c$$

$$= -\frac{3}{32}(2x-3)^{-4} - \frac{7}{10}(2x-3)^{-5} - \frac{41}{48}(2x-3)^{-6}+c.$$

(*iii*) Put $7\sin x+2=t.$

$\therefore \quad 7\cos dx=dt$

$\therefore \quad \cos x\,dx=\frac{1}{7}dt$

Also, $\sin x = \frac{t-2}{7}$

$\therefore \quad 4\sin x-5=4\left(\frac{t-2}{7}\right)-5$

$$= \frac{4t}{7}-\frac{43}{7}.$$

$$\therefore \int \frac{4\sin x-5}{(7\sin x+2)^8}\cos x$$

$$= \int\left(\frac{4t}{7}-\frac{43}{7}\right)t^{-8}\left(\frac{1}{7}\right)dt$$

$$= \frac{4}{49}\int t^{-7}\,dt - \frac{43}{49}\int t^{-8}\,dt$$

$$= \frac{4}{49}\left(\frac{t^{-6}}{-6}\right)-\frac{43}{49}\left(\frac{t^{-7}}{-7}\right)+c$$

$$= -\frac{2}{147}(7\sin x+2)^{-6}+\frac{43}{343}(7\sin x+2)^{-7}+c.$$

Note. Though the example (*iii*) given above does not belong to this class, it can be converted into this class and hence has been discussed above.

EXERCISE 5·5 (c)

Integrate the following with respect to x :

1. (*i*) $(2x+1)\sqrt{x-4}$; (*ii*) $(7x+5)(x-2)^{3/2}$

(*iii*) $(4-5x)(x-3)^{-5/2}$; (*iv*) $(9-4x)(x-1)^8$;

(*v*) $(12x-7)(4-x)^{7/2}$.

2. (*i*) $(3x+2)\sqrt{2x-1}$; (*ii*) $(7x-4)(3x+1)^{-3/2}$

(*iii*) $(6-3x)(2-3x)^{-1/2}$.

3. (*i*) $(2x^2+5x-1)\sqrt{x-2}$ (*ii*) $(3x^2-7x+4)\sqrt{x-1}$:
(*iii*) $(5x^2+4x+7)(2x+3)^{-3/2}$.

4. (*i*) $(2\tan x-3)^5(3\tan x+4)\sec^2 x$.

ANSWERS

1. (*i*) $\frac{4}{5}(x-4)^{5/2}+6(x-4)^{3/2}+c$.

(*ii*) $2(x-2)^{7/2}+\frac{38}{5}(x-2)^{5/2}+c$.

(*iii*) $10(x-3)^{-1/2}+\frac{22}{3}(x-3)^{-3/2}+c$.,

(*iv*) $-\frac{2}{5}(x-1)^{10}+\frac{5}{9}(x-9)^9+c$;

(*v*) $\frac{24}{11}(4-x)^{11/2}-\frac{88}{9}(4-x)^{9/2}+c$;

2. (*i*) $\frac{3}{10}(2x-1)^{5/2}+\frac{7}{6}(2x-1)^{3/2}+c$.,

(*ii*) $\frac{14}{9}(3x+1)^{1/2}+\frac{38}{9}(3x+1)^{-1/2}+c$.;

(*iii*) $-2(2-3x)^{1/2}-\frac{2}{9}(x-2)^{3/2}+c$.

3. (*i*) $\frac{4}{7}(x-2)^{7/2}+\frac{26}{5}(x-2)^{5/2}+\frac{34}{3}(x-2)^{3/2}+c$.

(*ii*) $\frac{6}{7}(x-1)^{7/2}-\frac{2}{5}(x-1)^{5/2}+c$.,

(*iii*) $\frac{5}{12}(2x+3)^{3/2}-\frac{11}{2}(2x+3)^{1/2}-\frac{49}{4}(2x+3)^{-1/2}+$.

IV. Integral of $\frac{f'(x)}{f(x)}$.

Let $$I=\int\frac{f'(x)}{f(x)}dx.$$

Put $$f(x)=t.$$

$\therefore$ $$f'(x)\,dx=dt.$$

$\therefore$ $$I=\int\frac{1}{t}\,dt=\log\,|\,t\,|+c.$$

$$=\log\,|\,f(x)\,|+c.$$

Illustrative Examples.

Ex. 1. Integrate the following with respect to x :

(*i*) $\frac{\sin 2x}{3+\sin^2 x}$; (*ii*) $\frac{1}{(1+x^2)\tan^{-1}x}$; (*iii*) $\frac{e^x-e^{-x}}{e^x+e^{-x}}$

Sol. Let $$I=\int\frac{\sin 2x}{3+\sin^2 x}\,dx.$$

Put $(3+\sin^2 x)=t.$

$\therefore\ 2\sin x\cos x\,dx=dt.$

$\therefore\quad \sin 2x\,dx=dt.$

$\because\quad I=\int\frac{1}{t}\,dt=\log\,|\,t\,|+c.$

$=\log\,|\,3+\sin^2 x\,|+c.$

(ii) Let $I=\int\frac{1}{\tan^{-1}x}\,\frac{1}{(1+x^2)}\,dx.$

Put $\tan^{-1}x=t.$

$\therefore\quad \frac{1}{1+x^2}\,dx=dt$

$\therefore\quad I=\int\frac{1}{t}\,dt=\log\,|\,t\,|+c.$

$=\log\,|\,\tan^{-1}x\,|+c.$

(iii) Let $I=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx.$

Put $e^x+e^{-x}=t.$

$\therefore\quad (e^x-e^{-x})dx=dt$

$\therefore\ I=\int\frac{1}{t}\,dt=\log\,|\,t\,|+c$

$=\log\,|\,e^x+e^{-x}\,|+c.$

EXERCISES 5·5 (d)

Integrate the following with respect to x.

1. $\frac{a\cos x-b\sin x}{a\sin x+b\cos x}$;
2. $\frac{1}{\sqrt{1-x^2}\sin^{-1}x}$;
3. $\frac{\sin x}{a+b\cos x}$
4. $\frac{\sin 2x}{a\cos^2 x+b\sin^2 x}$;
5. $\frac{\tan x\sec^2 x}{\sec^2 x+\tan^2 x}$;
6. $\frac{\cot x}{\log(\sin x)}$;
7. $\frac{\cos 2x}{(\cos x+\sin x)^2}$;
8. $\frac{e^{x-1}+x^{e-1}}{e^x+x^e}$;
9. $\frac{e^{2x}+1}{e^{2x}-1}$;
10. $\frac{1-\tan x}{1+\tan x}$;
11. $\frac{1}{x+x\log x}$;
12. $\frac{1}{x\log x\,[\log(\log x)]}$

ANSWERS

1. $\log\,|\,a\sin x+b\cos x\,|+c.$;
2. $\log\,|\,\sin^{-1}x\,|+c$;
3. $-\frac{1}{b}\log\,|\,a+b\cos x\,|+c$
4. $\frac{1}{b-a}|\,a\cos^2 x+b\sin^2 x\,|+c$;
5. $\frac{1}{2}\log-\sec^2 x+\tan^2 x\,|+c$;
6. $\log\,|\,\log\sin x\,|+c$;

7. $\log|\cos x+\sin|xe;+c$: **8.** $1/e\log|e^x+x^e|+c$

9. $\log|e^x-e^{-x}|+c$: **10.** $\log|\sin x+\cos x|+c$;

11. $\log|1+\log x|+c$; **12.** $\log|\log(\log x)|+c$.

V. Integrals of tan x, cot x, sec x and cosec x.

(*i*) $\int\tan x\,dx=\int\frac{\sin x}{\cos x}dx$

$$=\int-\frac{\frac{d}{dx}(\cos x)}{\cos x}dx$$

$$=-\log|\cos x|+c.$$

$$=\log|\sec x|+c.$$

$$\therefore\quad\int\tan x\,dx=\log|\sec x|+c.$$

(*ii*) $\int\cot x\,dx=\int\frac{\cos x}{\sin x}dx$

$$=\int\frac{\frac{d}{dx}(\sin x)}{\sin x}dx$$

$$=\log|\sin x|+c$$

$$\therefore\quad\int\cot x\,dx=\log|\sin x|+c$$

(*iii*) Let $I=\int\sec x\,dx=\int\frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)}dx$

$$=\int\frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx.$$

Now, put $\sec x+\tan x=t$.

$\therefore(\sec x\tan x+\sec^2 x)\,dx=dt.$

$$\therefore\quad I=\int\frac{1}{t}dt=\log|t|+c.$$

$$=\log|\sec x+\tan x|+c.$$

$$\therefore\quad\int\sec x\,dx=\log|\sec x+\tan x|+c.$$

(*iv*) Let $I=\int\operatorname{cosec}x\,dx=\int\frac{\operatorname{cosec}x(\operatorname{cosec}x+\cot x)}{\operatorname{cosec}x+\cot x}dx.$

$$=\int\frac{\operatorname{cosec}^2 x+\operatorname{cosec}x\cot x}{\operatorname{cosec}x+\cot x}dx.$$

Put $\operatorname{cosec}x+\cot x=t.$

$$\therefore\quad-(\operatorname{cosec}x\cot x+\operatorname{cosec}^2 x)\,dx=dt.$$

$$\therefore \quad I=\int -\frac{1}{t}\,dt=-\int \frac{1}{t}dt$$

$$=-\log \mid \operatorname{cosec} x+\cot x \mid +c$$

$$\therefore \int \operatorname{cosec} x\,dx=-\log \mid \operatorname{cosec} x+\cot x \mid +c.$$

Now, we add these formulae to the list of standard formulae.

27. $\int \tan x\,dx=\log \mid \sec x \mid +c.$

28. $\int \cot x\,dx=\log \mid \sin x \mid +c.$

29. $\int \sec x\,dx=\log \mid \sec x+\tan x \mid +c.$

30. $\int \operatorname{cosec} x\,dx=-\log \mid \operatorname{cosec} x+\cot x \mid +c.$

Illustrative Examples.

Ex. 1. Obtain the integrals of

$$(i)\ \frac{\cos 2x}{\sin x}\ ;\ (ii)\ \frac{\sin x}{\sqrt{1-\sin x}}\ ;\ (iii)\ \frac{1}{\sin\,(x-a)\,\sin(x-b)}$$

Sol. $(i)\ \int \frac{\cos 2x}{\sin x}\,dx=\int \frac{1-2\sin^2 x}{\sin x}\,dx$

$$=\int(\operatorname{cosec} x-2\sin x)\,dx$$

$$=-\log \mid \operatorname{cosec} x+\cot x \mid +2\cos x+c.$$

(ii) Let $I=\int \frac{\sin x}{\sqrt{1-\sin x}}=-\int \frac{1-\sin x-1}{\sqrt{(1-\sin x}}dx$

$$=-\int \sqrt{1-\sin x}+\int \frac{1}{\sqrt{1-\sin x}}dx$$

Now, $\quad =\int \sqrt{1-\sin x}\,dx=\int\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)dx$

$$=-2\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right).$$

Also, $\int \frac{1}{\sqrt{1-\sin x}}\,dx=\int \frac{1}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}dx$

$$=\frac{1}{\sqrt{2}}\int \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\cos\frac{x}{2}-\left(\frac{1}{\sqrt{2}}\right)\sin x}dx$$

$$=\frac{1}{\sqrt{2}}\int \frac{1}{\cos\frac{x}{4}\cos\frac{\pi}{4}-\sin x\,\sin\frac{\pi}{4}}\,dx$$

$$=\int \frac{1}{\cos\,(x+\pi/4)}dx=\int \sec\left(x+\frac{\pi}{4}\right)dx$$

$$= \log\left|\sec\left(x+\frac{\pi}{4}\right)+\tan\left(x+\frac{\pi}{4}\right)\right|+c.$$

$$\therefore \quad \int \frac{\sin x}{\sqrt{1-\sin x}}\,dx = \int \sqrt{1-\sin x}\,dx + \int \frac{1}{\sqrt{1-\sin x}}\,dx$$

$$= 2\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)+\log\left|\sec\left(x+\frac{\pi}{4}\right)+\tan\left(x+\frac{\pi}{4}\right)\right|+c.$$

(*iii*) $\int \frac{1}{\sin(x-a)\sin(x-b)}\,dx$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin[(x-b)-(x-a)]}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int [\cot(x-a)-\cot(x-b)]\,dx$$

$$= \frac{1}{\sin(a-b)}\log|\sin(x-a)|-\log\sin|x-b|+c.$$

$$= \frac{1}{\sin(a-b)}\log\left|\frac{\sin(x+a)}{\sin(x-b)}\right|+c.$$

EXERCISES 5·5 (e).

Integrate the following with respect to x

1. (*i*) $\tan(5-3x)$; (*ii*) $\cot 5x$; (*iii*) $\sec(3x+4)$; (*iv*) $\operatorname{cosec}(4x)$

2. (*i*) $x^2\cot x^3$ (*ii*) $\frac{\sec(\log x)}{x}$; (*iii*) $e^x\tan e^x$

3. (*i*) $\frac{\cos 2x}{\cos x}$; (*ii*) $\frac{\tan 2x}{\tan x}$

4. (*i*) $\frac{1}{\sqrt{1+\sin x}}$; (*ii*) $\frac{\sin x}{\sqrt{1+\sin x}}$

(*iii*) $\frac{1}{\sqrt{1+\cos 2x}}$; (*iv*) $\frac{1}{\sqrt{1-\cos x}}$.

5. (*i*) $\frac{1}{\cos(x-a)\cos(x-b)}$; (*ii*) $\frac{\cos x}{\cos(x+a)}$

(*iii*) $\frac{\sin(x-a)}{\sin(x+a)}$.

ANSWERS

1. (*i*) $\frac{1}{3}\log|\cos(5-8x)|+c$; (*ii*) $\frac{1}{6}\log|\sin 6x|+c$;
(*iii*) $\frac{1}{3}\log|\sec(3x+4)+\tan(3x+4)|+c$;
(*iv*) $-\frac{1}{4}\log|\operatorname{cosec} 4x+\cot 4x|+c$.

2. (*i*) $\frac{1}{3}\log|\sin x^3|+c$;
(*ii*) $\log|\sec(\log x)+\tan(\log x)|+c$;
(*iii*) $\log|\sec e^x|+c$.

3. (*i*) $2\sin x-\log|\sec x+\tan x|+c$;
(*ii*) $x+\log|\tan 2x+\sec 2x|+c$.

4. (*i*) $\sqrt{2}\log|\tan x+\sec x|+c$;

(*ii*) $2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)-\sqrt{2}\log\left|\tan\frac{x}{2}+\sec\frac{x}{2}\right|+c$;

(*iii*) $\frac{+1}{\sqrt{2}}\log\left|\operatorname{cosec}\left(\frac{\pi}{4}-x\right)+\cot\left(\frac{\pi}{4}-x\right)\right|+c$;

(*iv*) $\sqrt{2}\log\left|\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+c$.

5. (*i*) $\operatorname{cosec}(a-b)\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|+c$;

(*ii*) Hint : Put $\cos x=\cos[(x+a)-a]$ and expand ;
$x\cos a+\sin a.\log|\sec(x+a)|+c$;

(*iii*) $x\cos 2a-\sin a-\sin 2a.\log|\sin(x+a)|+c$.

VI. Standard Substitutions

The integrals which are of the following types or which contain a factor of the type, can be brought to a simpler form ; and sometimes directly to the standard form, by the following substitutions.

	Integral	Substitution
(*i*)	$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $x=a\cos\theta$.
(*ii*)	$\sqrt{a^2-x^2}$	$x=a\tan\theta$ or $x=a\cot\theta$.
(*iii*)	$\sqrt{x^2-a^2}$	$x=a\sec\theta$ or $x=a\operatorname{cosec}\theta$.
(*iv*)	$\sqrt{\frac{a-x}{a+x}}$	$x=a\cos 2\theta$.
(*v*)	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2=a^2\cos 2\theta$.

Illustrative Examples

Find the antiderivatives of

(*i*) $\frac{\sqrt{a^2-x^2}}{x^2}$; (*ii*) $x\sqrt{\frac{a^2-x^2}{a^2+x^2}}$

Sol. (*i*) Let $\mathrm{I}=\int\sqrt{\frac{a^2-x^2}{x^2}}\,dx$.

Put $x = a \sin\theta$.

$\therefore$ $dx = a\cos\theta\, d\theta$.

$\therefore$ $$I = \int \frac{\sqrt{a^2 - a^2\sin^2\theta}}{a^2\sin^2\theta} \cdot a\cos\theta\, d\theta$$

$$= \frac{a\cos\theta . a\cos\theta}{a^2\sin^2\theta}\, d\theta$$

$$= \int \cot^2\theta\, d\theta = \int (\operatorname{cosec}^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + c.$$

$$= \frac{-\cos\theta}{\sin\theta} - \theta + c$$

$$= -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a} + c.$$

(*ii*) Let $$I = \int x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\, dx$$

Put $x^2 = a^2\cos 2\theta$.

$\therefore$ $2x\, dx = -2a^2\sin 2\theta\, d\theta$.

$\therefore$ $$I = \int \sqrt{\frac{a^2 - a^2\cos 2\theta}{a^2 + a^2\cos 2\theta}}\, (-a^2\sin 2\theta)\, d\theta$$

$$= -a^2 \int \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}} . 2\sin\theta\cos\theta\, d\theta$$

$$= -a^2 \int \frac{\sin\theta}{\cos\theta} . 2\sin\theta\cos\theta\, d\theta$$

$$= -a^2 \int 2\sin^2\theta\, d\theta = -a^2 \int (1 - \cos 2\theta)\, d\theta$$

$$= -a^2 \left(\theta - \frac{\sin 2\theta}{2}\right) + c.$$

$$= -a^2 \left[\tfrac{1}{2}\cos^{-1}\left(\frac{x^2}{2}\right) - \tfrac{1}{2}\frac{\sqrt{a^4 - x^4}}{a^2}\right] + c$$

$$\left(\because \cos 2\theta = \frac{x^2}{a^2}\right)$$

$$= \frac{1}{2}\left[\sqrt{a^4 - x^4} - a^2\cos^{-1}\left(\frac{x^2}{a^2}\right)\right] + c.$$

EXERCISE 5·5. (*f*)

Integrate the following.

1. $\dfrac{1}{x^2\sqrt{x^2 - 1}}$
2. $\dfrac{1}{x^2\sqrt{1 + x^2}}$
3. $\dfrac{1}{(x^2 + a^2)^2}$
4. $\sqrt{\dfrac{a - x}{a + x}}$

5. $\sqrt{\frac{a-x}{x}}$ 6. $\frac{1}{(2ax+x^2)^{3/2}}$

ANSWERS

1. $\frac{\sqrt{x^2-1}}{x}+c$ 2. $-\frac{\sqrt{1+x^2}}{x}+c$

3. $\frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)+\frac{1}{2a^2}\cdot\frac{x}{x^2+a^2}+c$

4. $\sqrt{a^2-x^2}-a\cos^{-1}\left(\frac{x}{a}\right)+c$;

5. Hint : Put $x=a\sin^2\theta$; $a\sin^{-1}\left(\sqrt{\frac{x}{a}}\right)+\sqrt{x(a-x)}+c$.

6. $-\frac{1}{a^2}\frac{a+x}{\sqrt{2ax+x^2}}$.

VII. Integrals of $\frac{1}{\sqrt{a^2-x^2}}$; $\frac{1}{x^2+a^2}$; $\frac{1}{\sqrt{x^2-a^2}}$; $\frac{1}{\sqrt{x^2+a^2}}$ $\frac{1}{x\sqrt{x^2-a^2}}$

Here we clearly see that the standard substitutions are useful in finding the integrals of the above functions.

(*i*) Let $I=\int\frac{dx}{\sqrt{a^2-x^2}}$

Put $x=a\sin\theta$

$\therefore$ $dx=a\cos\theta\,d\theta$ and $\theta=\sin^{-1}\left(\frac{x}{a}\right)$.

$\therefore$ $I=\int\frac{a\cos\theta\,d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$

$=\int\frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}\,d\theta=\int 1\,d\theta$

$=\sin^{-1}\left(\frac{x}{a}\right)+c.$

(*ii*) Let $I=\int\frac{dx}{x^2+a^2}$.

Put $x=a\tan\theta$

$\therefore$ $dx=a\sec^2\theta\,d\theta$ and $\theta=\tan^{-1}\left(\frac{x}{a}\right)$

$\therefore$ $I=\int\frac{a\sec^2\theta}{a^2\tan^2\theta+a^2}\,d\theta$

$=\int\frac{a\sec^2\theta}{a^2(\tan^2\theta+1)}\,d\theta$

$$= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{a}\int 1\,d\theta = \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.$$

(*iii*) Let $\quad I = \int \frac{dx}{\sqrt{x^2-a^2}}.$

Put $\quad x = a\sec\theta.$

$\therefore \quad dx = a\sec\theta\tan\theta\,d\theta$

$\therefore \quad I = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}\,d\theta$

$$= \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2(\sec^2\theta-1)}}\,d\theta$$

$$= \int \frac{\sec\theta\tan\theta}{\tan\theta}\,d\theta = \int \sec\theta\,d\theta$$

$$= \log|\sec\theta + \tan\theta| + c$$

$$= \log|\sec\theta + \sqrt{\sec^2\theta - 1}| + c_1$$

$$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + c_1$$

$$= \log\left|\frac{x + \sqrt{x^2-a^2}}{a}\right| + c_1.$$

$$= \log|x + \sqrt{x^2-a^2}| - \log|a| + c_1$$

$$= \log|x + \sqrt{x^2-a^2}| + c$$

where $\quad c = -\log|a| + c_1.$

(*iv*) Let $\quad I = \int \frac{dx}{\sqrt{x^2+a^2}}$

Put $\quad x = a\tan\theta.$

$\therefore \quad dx = a\sec^2\theta\,d\theta.$

$\therefore \quad I = \int \frac{a\sec^2\theta}{\sqrt{a^2\tan^2\theta + a^2}}\,d\theta$

$$= \int \frac{a\sec^2\theta}{\sqrt{a^2(\tan^2\theta+1)}}\,d\theta$$

$$= \int \frac{a\sec^2\theta}{a\sec\theta}\,d\theta = \int \sec\theta\,d\theta$$

$$= \log|\sec\theta + \tan\theta| + c_1$$

$$=\log|\sqrt{\tan^2\theta+1}+\tan\theta|+c_1$$

$$=\log\left|\sqrt{\frac{x^2}{a^2}+1}+\frac{x}{a}\right|+c_1$$

$$=\log\left|\frac{x+\sqrt{x^2+a^2}}{a}\right|+c_1$$

$$=\log|x+\sqrt{x^2+a^2}|-\log|a|+c_1$$

$$=\log|x+\sqrt{x^2+a^2}|+c,$$

where $c=\log|a|+c_1$.

(*v*) Let $I=\int\frac{dx}{x\sqrt{x^2-a^2}}$.

Put $x=a\sec\theta$.

$\therefore$ $dx=a\sec\theta\tan\theta\,d\theta$

and $\theta=\sec^{-1}\left(\frac{x}{a}\right)$.

$\therefore$ $$I=\int\frac{a\sec\theta\tan\theta}{a\sec\theta\sqrt{a^2\sec^2\theta-a^2}}d\theta$$

$$=\int\frac{a\sec\theta\tan\theta}{a\sec\theta\sqrt{a^2(\sec^2\theta-1)}}d\theta$$

$$=\frac{1}{a}\int\frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\,d\theta$$

$$=\frac{1}{a}\int 1\,d\theta=\frac{1}{a}\theta+c$$

$$=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c.$$

Now we can add to our list of standard formulae the above results.

31. $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+c$;

32. $\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c.$

33. $\int\frac{dx}{\sqrt{x^2-a^2}}=\log|x+\sqrt{x^2-a^2}|+c.$

34. $\int\frac{dx}{\sqrt{x^2+a^2}}=\log|x+\sqrt{x^2+a^2}|+c.$

35. $\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c.$

The following examples will make the procedure clear.

Illustrative Examples

Ex. 1. Obtain the integrals of

(*i*) $\dfrac{1}{\sqrt{9-4x^2}}$ (*ii*) $\dfrac{1}{9+4x^2}$

Sol. In the examples of this type, we make the coefficient of x^2 as 1 and proceed.

(*i*) $$\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{3/2}\right) + c.$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c.$$

(*ii*) $$\int \frac{dx}{9+4x^2} = \frac{1}{4}\int \frac{dx}{\left(\frac{3}{2}\right)^2 + x^2}$$

$$= \frac{1}{4}\left[\frac{2}{3}\tan^{-1}\left[\frac{x}{\frac{3}{2}}\right]\right] + c.$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c.$$

Ex. 2. Obtain the integrals of

(*i*) $\dfrac{1}{\sqrt{3x^2-7}}$; (*ii*) $\dfrac{1}{\sqrt{3x^2+7}}$

Sol. (*i*) $$\int \frac{dx}{\sqrt{3x^2-7}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{x^2 - \frac{7}{3}}}$$

$$= \frac{1}{\sqrt{3}}\log\left|x + \sqrt{x^2 - \frac{7}{3}}\right| + c_1$$

$$= \frac{1}{\sqrt{3}}\log\left|x + \frac{\sqrt{3x^2-1}}{\sqrt{3}}\right| + c_1$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{x\sqrt{3} + \sqrt{3x^2-7}}{\sqrt{3}}\right| + c_1$$

$$= \frac{1}{\sqrt{3}} \log | x\sqrt{3} + \sqrt{3x^2-7} |$$

$$- \frac{1}{\sqrt{3}} \log | \sqrt{3} + c_1$$

$$= \frac{1}{\sqrt{3}} \log | x\sqrt{3} + \sqrt{3x^2-7} | + c$$

where $c = -\frac{1}{\sqrt{3}} \log \sqrt{3} + c_1$

(*ii*) $$\int \frac{dx}{\sqrt{3x^2+7}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{x^2+\frac{7}{3}}}$$

$$= \frac{1}{\sqrt{3}} \log \left| x + \sqrt{x^2+\frac{7}{3}} \right| + c_1$$

$$= \frac{1}{\sqrt{3}} \log \left| x + \frac{\sqrt{3x^2+7}}{\sqrt{3}} \right| + c_1$$

$$= \frac{1}{\sqrt{3}} \log \left| \frac{x\sqrt{3} + \sqrt{3x^2+7}}{\sqrt{3}} \right| + c_1$$

$$= \frac{1}{\sqrt{3}} \log \left| x\sqrt{3} + \sqrt{3x^2+7} \right|$$

$$- \frac{1}{\sqrt{3}} \log \sqrt{3} + c_1$$

$$= \frac{1}{\sqrt{3}} \log \left| x\sqrt{3} + \sqrt{3x^2+7} \right| + c$$

where $c = -\frac{1}{\sqrt{3}} \log \sqrt{3} + c_1$

Ex. 3. Obtain the integrals of :

(*i*) $\frac{1}{\sqrt{5+4x-x^2}}$; (*ii*) $\frac{1}{x^2-4x+13}$

Sol. In the examples of the above type, we express the denominator either as x^2+a^2 or x^2-a^2 or a^2-x^2 and proceed.

(*i*) $$\int \frac{dx}{\sqrt{5+4x-x^2}} = \int \frac{dx}{\sqrt{9-4+4x-x^2}}$$

$$= \int \frac{dx}{\sqrt{3^2-(x-2)^2}}$$

$$= \sin^{-1} \left(\frac{x-2}{3} \right) + c.$$

$$(ii)\quad \int \frac{dx}{x^2-4x+13} = \int \frac{dx}{x^2-4x+4+9} = \int \frac{dx}{(x-2)^2+3^2}$$
$$= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+c.$$

Ex. 4. Obtain the integrals of :

$$(i)\quad \frac{1}{\sqrt{2x^2+7x-5}}\,; \qquad (ii)\quad \frac{1}{\sqrt{2x^2+7x+13}}$$

Sol. In the examples of the above type we first make the coefficient of x^2 as 1 and express the denominator as x^2+a^2 as x^2-a^2 or a^2-x^2 and proceed.

$$(i)\quad \int \frac{dx}{\sqrt{2x^2+7x-5}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+\frac{7x}{2}-\frac{5}{2}}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+\frac{7x}{2}+\frac{49}{16}-\frac{89}{16}}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x+\frac{7}{2}\right)^2-\left(\frac{\sqrt{89}}{4}\right)^2}}$$
$$= \frac{1}{\sqrt{2}}\log\left|\left(x+\frac{7}{4}\right)+\sqrt{\left(x+\frac{7}{4}\right)^2-\left(\frac{\sqrt{89}}{4}\right)^2}\right|+c$$
$$= \frac{1}{\sqrt{2}}\log\left|\left(x+\frac{7}{4}\right)+\sqrt{x^2+\frac{7x}{2}-\frac{5}{2}}\right|+c$$

$$(ii)\quad \int \frac{dx}{\sqrt{2x^2+7x+13}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x+\frac{7}{4}\right)^2+\left(\frac{\sqrt{55}}{4}\right)^2}}$$
$$= \frac{1}{\sqrt{2}}\log\left|\left(x+\frac{7}{4}\right)+\sqrt{\left(x+\frac{7}{4}\right)^2+\left(\frac{\sqrt{55}}{4}\right)^2}\right|+c^2$$
$$= \frac{1}{\sqrt{2}}\log\left|\left(x+\frac{7}{4}\right)+\sqrt{x^2+\frac{7x}{2}+\frac{13}{2}}\right|+c$$

Ex. 5. Find $\int \sqrt{\frac{a+x}{a-x}}\,dx$

Sol. Let $$I = \int \sqrt{\frac{a+x}{a-x}}\,dx$$
$$= \int \frac{\sqrt{a+x}}{\sqrt{a-x}} \times \frac{\sqrt{a+x}}{\sqrt{a+x}}\,dx$$

$$= \int \frac{a+x}{\sqrt{a^2-x^2}}\, dx$$

$$= \int \frac{a}{\sqrt{a^2-x^2}}\, dx + \int \frac{x}{\sqrt{a^2-x^2}}\, dx$$

$$= a \int \frac{dx}{\sqrt{a^2-x^2}} + \int \frac{x}{\sqrt{a^2-x^2}}\, dx.$$

Now $$I_1 = a\int \frac{dx}{\sqrt{a^2-x^2}} = a \sin^{-1}\left(\frac{x}{a}\right).$$

Let $$I_2 = \int \frac{x}{\sqrt{a^2-x^2}}\, dx$$

Put $x = a \sin\theta \quad \therefore \quad dx = a\cos\theta\, d\theta.$

$\therefore$ $$I_2 = \int \frac{a \sin\theta}{\sqrt{a^2 - a^2\sin^2\theta}}\, a\cos\theta\, d\theta$$

$$= \int \frac{a\sin\theta}{a\cos\theta}\, a\cos\theta\, d\theta.$$

$$= a \int \sin\theta\, d\theta = -a\cos\theta$$

$$= -a\sqrt{1-\sin^2\theta}$$

$$= -a\sqrt{1-\frac{x^2}{a^2}} = -a\,\frac{\sqrt{a^2-x^2}}{a}$$

$$= -\sqrt{a^2-x^2}$$

$$\therefore \int \sqrt{\frac{a+x}{a-x}}\, dx = a \sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2} + c.$$

Ex. 6. Evaluate $\dfrac{\cos x\, dx}{\sqrt{\sin^2 x + 2\sin x - 3}}$

Sol. Put $\sin x = t. \quad \therefore \quad \cos x\, dx = dt.$

$$\therefore \int \frac{\cos x\, dx}{\sqrt{\sin^2 x + 2\sin x - 3}} = \int \frac{dt}{\sqrt{t^2+2t+3}} = \int \frac{dt}{\sqrt{t^2+2t+1-4}}$$

$$= \int \frac{dt}{\sqrt{(t+1)^2-(2)^2}} = \log\left[(t+1) + \sqrt{(t+1)^2-(2)^2}\right] + c.$$

$$= \log\left[(t+1) + \sqrt{t^2+2t-3}\right] + c.$$

$$= \log\left[\sin x + 1) + \sqrt{\sin^2 + 2\sin x - 3}\right] + c.$$

EXERCISE 5·5. (*g*)

Obtain the integrals of :

1. (*i*) $\frac{1}{\sqrt{16-x^2}}$; (*ii*) $\frac{1}{x^2+16}$

(*iii*) $\frac{1}{\sqrt{x^2-16}}$; (*iv*) $\frac{1}{\sqrt{16+x^2}}$.

2. (*i*) $\frac{1}{\sqrt{7-x^2}}$; (*ii*) $\frac{1}{x^2+7}$;

(*iii*) $\frac{1}{\sqrt{x^2-7}}$; (*iv*) $\frac{1}{\sqrt{x^2+7}}$.

3. (*i*) $\frac{1}{\sqrt{16-9x^2}}$; (*ii*) $\frac{1}{9x^2+16}$;

(*iii*) $\frac{1}{\sqrt{9x^2-16}}$; (*iv*) $\frac{1}{\sqrt{9x^2+16}}$.

4. (*i*) $\frac{1}{\sqrt{17-5x^2}}$; (*ii*) $\frac{1}{5x^2+17}$;

(*iii*) $\frac{1}{\sqrt{5x^2-17}}$; (*iv*) $\frac{1}{\sqrt{5x^2+17}}$.

5. (*i*) $\frac{1}{\sqrt{16-x^2-6x}}$; (*ii*) $\frac{1}{x^2+6x+34}$;

(*iii*) $\frac{1}{\sqrt{x^2+6x-16}}$; (*iv*) $\frac{1}{\sqrt{x^2+6x+34}}$.

6. (*i*) $\frac{1}{\sqrt{7-3x^2+5x}}$; (*ii*) $\frac{1}{3x^2+5x+7}$;

(*iii*) $\frac{1}{\sqrt{3x^2-5x-7}}$; (*iv*) $\frac{1}{\sqrt{3x^2+5x+7}}$.

7. (*i*) $\frac{\sin x}{\sqrt{1+3\cos^2 x}}$; (*ii*) $\frac{\sec^2\theta}{5\tan^2\theta-12\tan\theta+14}$;

(*iii*) $\frac{x^2}{\sqrt{x^6+2x^3+3}}$.

8. (*i*) $\frac{e^x}{\sqrt{e^{2x}-1}}$; (*ii*) $\frac{e^x}{e^{2x}+16}$;

(*iii*) $\frac{1}{e^{-2x}+25}$.

9. (*i*) $\frac{1}{\sqrt{1-x^2}[9+(\sin^{-1}x)^2]}$; (*ii*) $\frac{1}{(1+x^2)\sqrt{3[\tan^{-1}x]^2+\tan^{-1}x+4}}$

ANSWERS

1. (*i*) $\sin^{-1}\left(\frac{x}{4}\right)+c$; (*ii*) $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+c$;

(*iii*) $\log|x+\sqrt{x^2-16}|+c$; (*iv*) $\log|x+\sqrt{x^2+16}|+c$.

2. (*i*) $\sin^{-1}\left(\frac{x}{\sqrt{7}}\right)+c$; (*ii*) $\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right)+c$;

(*iii*) $\log|x+\sqrt{x^2-7}|+c$; (*iv*) $\log|x+\sqrt{x^2+7}|+c$.

3. (*i*) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right)+c$; (*ii*) $\frac{1}{12}\tan^{-1}\left(\frac{3x}{4}\right)+c$;

(*iii*) $\frac{1}{3}\log|3x+\sqrt{9x^2-16}|+c$;

(*iv*) $\frac{1}{3}\log|3x+\sqrt{9x^2+16}|+c$.

4. (*i*) $\frac{1}{\sqrt{5}}\sin^{-1}\left(\frac{x\sqrt{5}}{\sqrt{17}}\right)+c$; (*ii*) $\frac{1}{\sqrt{85}}\tan^{-1}\frac{x\sqrt{5}}{\sqrt{17}}+c$;

(*iii*) $\frac{1}{\sqrt{5}}\log\left|x\sqrt{5}+\sqrt{5x^2-17}\right|+c$;

(*iv*) $\frac{1}{\sqrt{5}}\log\left|x\sqrt{5}+\sqrt{5x^2+17}\right|+c$.

5. (*i*) $\sin^{-1}\left(\frac{x+3}{5}\right)+c$; (*ii*) $\frac{1}{5}\tan^{-1}\left(\frac{x+3}{5}\right)+c$;

(*iii*) $\log|(x+3)+\sqrt{x^2+6x-16}|+c$;

(*iv*) $\log|(x+3)+\sqrt{x^2+6x+34}|+c$.

6. (*i*) $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{6x-5}{\sqrt{109}}\right)+c$.

(*ii*) $\frac{2}{\sqrt{59}}\tan^{-1}\left(\frac{6x+5}{\sqrt{59}}\right)+c$;

(*iii*) $\frac{1}{\sqrt{3}}\log\left|\left(x+\frac{5}{6}\right)+\sqrt{x^2+\frac{5x}{3}-\frac{7}{3}}\right|+c$;

(*iv*) $\frac{1}{\sqrt{3}}\log\left|x+\frac{5}{6}+\sqrt{x^2+\frac{5x}{3}+\frac{7}{3}}\right|+c$.

7. (*i*) $\frac{-1}{\sqrt{3}}\log(\sqrt{3}\cos x+\sqrt{1+3\cos^2 x})+c$;

(*ii*) $\frac{\sqrt{5}}{\sqrt{34}}\tan^{-1}\left[\frac{5\tan\theta-6}{\sqrt{34}}\right]+c$;

(*iii*) Hint : Put $x^3=t$;

Ans. $\frac{1}{3}\log\left|(x^3+1)+\sqrt{x^6+2x^3+3}\right|+c$.

3. (*i*) $\log|e^x+\sqrt{e^{2x}-1}|+c$;

(*ii*) $\frac{1}{4}\tan^{-1}\left(\frac{e^x}{4}\right)+c$;

(*iii*) Hint : $\frac{1}{\sqrt{e^{-2x}+25}}=\frac{e^x}{\sqrt{1+25\,e^{2x}}}$;

Ans. $\frac{1}{5}\log\left|5e^x+\sqrt{1+25e^{2x}}\right|+c$;

9. (*i*) $\frac{1}{3}\tan^{-1}\left(\frac{\sin^{-1}x}{3}\right)+c$;

(*ii*) $\frac{1}{\sqrt{3}}\log\left|\left(\sqrt{3}\tan^{-1}x+\frac{1}{\sqrt{3}}\right)+\sqrt{3(\tan^{-1}x)^2+2\tan^{-1}x+4}\right|+c$

VII. Integrals of $\dfrac{px+q}{ax^2+bx+c}$, $\dfrac{px+q}{\sqrt{ax^2+bx+c}}$, $\dfrac{f(x)}{ax^2+bx+c}$.

We observe that

$$\frac{d}{dx}(ax^2+bx+c)=2ax+b.$$

We express $px+q\equiv A(2ax+b)+B$ for suitable A and B and proceed. The following examples will make the procedure clear.

Ex. 1. Evaluate the following integrals :

(*i*) $\int\frac{5x+2}{3x^2+4x+5}dx$; (*ii*) $\int\frac{5x+2}{\sqrt{3x^2+4x+5}}dx$

Sol. (*i*) $\dfrac{d}{dx}(3x^2+4x+5)=6x+4$

Let $5x+2\equiv A(6x+4)+B$, for suitable A and B. Equating the coefficients of x and the constants, we get

$$6A=5 \text{ and } 4A+B=2.$$

Solving these equations, we get

$$A=\frac{5}{6} \text{ and } B=-\frac{4}{3}.$$

$$\therefore \quad 5x+2=\frac{5}{6}(6x+4)-\frac{4}{3}.$$

$$\therefore \quad \int\frac{5x+2}{3x^2+4x+5}dx=\int\frac{\frac{5}{6}(6x+4)-\frac{4}{3}}{3x^2+4x+5}dx$$

$$=\frac{5}{6}\int\frac{6x+4}{3x^2+4x+5}dx-\frac{4}{3}\int\frac{dx}{3x^2+4x+5}$$

Now,

$$\int\frac{6x+4}{3x^2+4x+5}dx=\int\frac{\frac{d}{dx}\left(3x^2+4x+5\right)}{3^2+4x+5}dx$$

$$=\log|3x^2+4x+5|.$$

Also,

$$\int\frac{1}{3x^2+4x+5}dx=\frac{1}{3}\int\frac{1}{x^2+\frac{4}{3}x+\frac{5}{3}}dx$$

$$= \frac{1}{3}\int \frac{1}{\left(x^2+\frac{4}{3}x+\frac{4}{9}\right)+\frac{11}{9}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{11}}{3}\right)^2}\,dx$$

$$= \frac{1}{3}\left(\frac{3}{\sqrt{11}}\right)\tan^{-1}\left(\frac{x+\frac{2}{3}}{\frac{\sqrt{11}}{3}}\right)$$

$$= \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{3x+2}{\sqrt{11}}\right)$$

$\therefore$ We get

$$\int \frac{5x+2}{3x^2+4x+5}\,dx = \frac{5}{6}\log\left|3x^2+4x+5\right| - \frac{4}{3\sqrt{11}}\tan^{-1}\left(\frac{3x+2}{\sqrt{11}}\right)+c.$$

(*ii*) As in example (*i*),

$$5x+2 = \frac{5}{6}(6x+4) - \frac{4}{3}.$$

$$\therefore \int \frac{5x+2}{\sqrt{3x^2+4+5}}\,dx = \int \frac{\frac{5}{6}(6x+4)-\frac{4}{3}}{\sqrt{3x^2+4x+5}}\,dx$$

$$= \frac{5}{6}\int \frac{6x+4}{\sqrt{3x^2+4x+5}}\,dx - \frac{4}{3}\int \frac{1}{\sqrt{3x^2+4x+5}}\,dx.$$

$$= \frac{5}{6}I_1 - \frac{4}{3}I_2 = (\text{say})$$

Now, in I_1 put

$$3x^2+4x+5 = t.$$

$$\therefore \quad (6x+4)dx = dt.$$

$$\therefore \quad I_1 = \int \frac{1}{t^{1/2}}\,dt = \int t^{-1/2}\,dt = 2t^{1/2} = 2(3x^2+4x+5)^{1/2}$$

$$= 2\sqrt{3x^2+4x+5}$$

Further,

$$I_2 = \int \frac{1}{\sqrt{3x^2+4x+5}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{11}}{2}\right)}}$$

$$= \frac{1}{\sqrt{3}} \log \left| \left(x + \frac{2}{3} \right) + \sqrt{x^2 + \frac{4x}{3} + \frac{5}{3}} \right|$$

$$\therefore \quad \int \frac{5x+2}{\sqrt{3x^2+4x+5}}\, dx$$

$$= \frac{5}{3} \sqrt{3x^2+4x+5} - \frac{4}{3\sqrt{3}} \log \left| \left(x + \frac{2}{3} \right) + \sqrt{x^2 + \frac{4x}{3} + \frac{5}{3}} \right|$$

Ex. 2. Evaluate the integral

$$\int \frac{2x^4 - 3x^3 - 4x + 3}{x^2 - 2x + 5}\, dx$$

Sol. When the degree of the polynomial in the numerator is greater than or equal to the degree of the polynomial in the denominater (*i.e.* greater than or equal to 2) ; we divide the numerator by the quadratic expression of the denominator and then proceed as follows.

$$2x^4 - 3x^3 - 4x + 3$$

$$= (2x^4 - 4x^3 + 10x^2) + (x^3 + 2x^2 + 5x) - (8x^2 - 16x + 40) - 25x + 43$$

$$= 2x^2(x^2 - 2x + 5) + x(x^2 - 2x + 5) - 8(x^2 - x + 5) - (25x - 43)$$

$$= (2x^2 + x - 8)(x^2 - 2x + 5) - (25x - 43).$$

$$\therefore \quad \int \frac{2x^4 - 3x^2 - 4x + 3}{x^2 - 2x + 5}\, dx$$

$$= \int \frac{(2x^2 + x - 8)(x^2 - 2x + 5) - (25x - 43)}{x^2 - 2x + 5}\, dx$$

$$= \int \left[2x^2 + x - 8 - \frac{25x - 43}{x^2 - 2x + 5} \right] dx$$

$$= \int (2x^2 + x - 8)\, dx - \int \frac{25x - 43}{x^2 - 2x + 5}\, dx$$

$$= \frac{2x^3}{3} + \frac{x^2}{2} - 8x - \int \frac{25x - 43}{x^2 - 2x + 5}\, dx$$

Now we evaluate $\int \frac{25x - 43}{x^2 - 2x + 5}\, dx$ as in Example 1.

$$\frac{d}{dx}(x^2 - 2x + 5) = 2x - 2.$$

Let $\quad 25x - 43 \equiv A(2x - 2) + B$

for some A and B.

$\therefore$ Equating the coefficients of x and the constant terms, we get

$$2A=25$$

and $$-2A+B=-43.$$

$$\therefore \quad A=\frac{25}{2}$$

and $$B=-18$$

$$\therefore \quad 25x-43=\frac{25}{2}(2x-2)+8$$

$$\therefore \quad \int \frac{25x-43}{x^2-2x+5}=\int \frac{\frac{25}{2}(2x-2)-18}{x^2-2x+5}dx$$

$$=\frac{25}{2}\int \frac{(2x-2)}{x^2-2x+5}dx-18\int \frac{1}{x^2-2x+5}dx$$

$$=\frac{25}{2}\log|x^2-2x+5|-18\int \frac{1}{(x-1)^2+(2)^2}dx$$

$$=\frac{25}{2}\log|x^2-2x+5|-18.\tfrac{1}{2}.\tan^{-1}\left(\frac{x-1}{1}\right)$$

$$=\frac{25}{2}\log|x^2-2x+5|-9\tan^{-1}\left(\frac{x-1}{2}\right).$$

$$\therefore \quad \int \frac{2x^4-3x^3-4x+3}{x^2-2x+5}dx$$

$$=\frac{2x^3}{2}+\frac{x^2}{2}-8x-\frac{25}{2}\log|x^2-2x+5|$$

$$+9\tan^{-1}\left(\frac{x-1}{2}\right)+c.$$

EXERCISE 5·5 (*h*)

Evaluate the integrals of the following functions:

1. (*i*) $\dfrac{2x-3}{2x^2+3x+7}$; (*ii*) $\dfrac{5x-1}{2x^2+3x+7}$;

   (*iii*) $\dfrac{3x+5}{4x^2-3x+1}$.

2. (*i*) $\dfrac{2x-3}{\sqrt{2x^2+3x+7}}$; (*ii*) $\dfrac{5x-1}{\sqrt{2x^2+3x+7}}$;

   (*iii*) $\dfrac{3x+5}{\sqrt{4x^2-3x+1}}$.

3. (*i*) $\dfrac{7x+3}{\sqrt{x^2-2x+3}}$; (*ii*) $\dfrac{7x+3}{\sqrt{x^2-2x-3}}$;

   (*iii*) $\dfrac{7x+3}{\sqrt{3+2x-x^2}}$.

4. (*i*) $\frac{2x^4-3x^2-2x+4}{x^2-2x+2}$; (*ii*) $\frac{3x^4-3x^3-x^2+1}{x^2+2x+2}$;

(*iii*) $\frac{3x^3+4x^2+12x-7}{x^2+2x+5}$.

5. Evaluate $\int \frac{(3\sin x+4)\cos x}{\sqrt{5\cos^2 x-12\sin x+4}}\,dx$

ANSWERS

1. (*i*) $\frac{1}{2}\log|2x^2+3x+7| - \frac{9}{\sqrt{47}}\tan^{-1}\left(\frac{4x+3}{\sqrt{47}}\right)+c$;

(*ii*) $\frac{5}{4}\log|2x^2+3x+7| - \frac{19}{\sqrt{47}}\tan^{-1}\left(\frac{4x+3}{\sqrt{47}}\right)+c$;

(*iii*) $\frac{3}{8}\log|4x^2-3x+1| + \frac{49}{\sqrt{47}}\tan^{-1}\left(\frac{8x-3}{\sqrt{47}}\right)+c$.

2. (*i*) $\sqrt{2x^2+3x-7}-\frac{9}{4}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3x}{2}+\frac{7}{2}}\right|+c$;

(*ii*) $\frac{5}{2}\sqrt{2x^2+3x-7}-\frac{19}{8}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3x}{2}+\frac{7}{2}}\right|+c$;

(*iii*) $\frac{3}{4}\sqrt{4x^2-3x+1}+\frac{49}{32}\log\left|\left(x-\frac{3}{8}\right)+\sqrt{x^2-\frac{3x}{4}+\frac{1}{4}}\right|+c$.

3. (*i*) $7\sqrt{x^2-2x+3}+10\log|(x-1)+\sqrt{x^2-2x+3}|+c$;
(*ii*) $7\sqrt{x^2-2x-3}+10\log|(x-1)+\sqrt{x^2-2x-3}|+c$.

(*iii*) $-7\sqrt{3+2x-x^2}+10\sin^{-1}\left(\frac{x-1}{4}\right)+c$.

4. (*i*) $\frac{2x^3}{2}+\frac{x^2}{2}-2x-4\log|x^2-2x+2|+c$;

(*ii*) $x^3-\frac{9x^2}{2}+11x-2\log|x^2+2x+2|-\frac{11}{2}\tan^{-1}(x+1)+c$;

(*iii*) $\frac{3x^2}{2}-2x+\frac{1}{2}\log|x^2+2x+5|+\tan^{-1}\left(\frac{x+1}{2}\right)+c$;

5. Hint. Express $5\cos^2 x-12\sin x+4$ in terms of only $\sin x$ and then put $t=\sin x$.

Ans: $-\frac{3}{5}\sqrt{5\cos^2 x-12\sin x+4}+\frac{2}{5\sqrt{5}}\sin^{-1}\left(\frac{5\sin x+6}{9}\right)+c$.

IX. Integrals of $\frac{1}{x^2-a^2}$, $\frac{1}{a^2-x^2}$.

(*i*) We can express

$$\frac{1}{x^2-a^2}=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right).$$

$$\therefore \int\frac{1}{x^2-a^2}\,dx=\frac{1}{2a}[\log|x-a|-\log|x+a|]+c;$$

$$=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c;$$

Here we have assumed that $x^2-a^2>0$.

(*ii*) In the same way above, if $x^2<a^2$, we have

$$\frac{1}{a^2-x^2}=\frac{1}{2a}\left(\frac{1}{a+x}+\frac{1}{a-x}\right)$$

Consequently, we get

$$\int\frac{1}{a^2-x^2}\,dx=\frac{1}{2a}\left[\int\frac{1}{a+x}\,dx+\int\frac{1}{a-x}\,dx.\right]$$

$$=\frac{1}{2a}\,[\log|a+x|-\log|a-x|]+c\,;$$

$$=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c\,;$$

Now, we add these results to our list of standard formulae :

36. $\displaystyle\int\frac{1}{x^2-a^2}\,dx=\frac{1}{2a}\log 2\left|\frac{x-a}{x+a}\right|+c.$

37. $\displaystyle\int\frac{1}{a^2-x^2}\,dx=\frac{1}{2a}\log 2\left|\frac{a+x}{a-x}\right|+c.$

EXERCISE 5·5. (i)

Obtain the integrals of the following:

1. (*i*) $\dfrac{1}{x^2-25}$; (*ii*) $\dfrac{1}{16-x^2}$;

(*iii*) $\dfrac{1}{7-x^2}$; (*iv*) $\dfrac{1}{x^2-15}$;

2. (*i*) $\dfrac{1}{4x^2-9}$; (*ii*) $\dfrac{1}{9-4x^2}$;

(*iii*) $\dfrac{1}{7-3x^2}$; (*iv*) $\dfrac{1}{7x^2-15}$;

3. (*i*) $\dfrac{1}{x^2-4x-21}$; (*ii*) $\dfrac{1}{21+4x-x^2}$;

(*iii*) $\dfrac{1}{x^2+6x+5}$;

4. (*i*) $\dfrac{1}{2x^2+3x-1}$; (*ii*) $\dfrac{1}{1-3x-2x^2}$;

5. (*i*) $\dfrac{2x+3}{2x^2+3x-1}$; (*ii*) $\dfrac{5x-1}{1-3x-2x^2}$;

ANSWERS

1. (*i*) $\dfrac{1}{10}\log\left|\dfrac{x-5}{x+5}\right|+c$; (*ii*) $\dfrac{1}{8}\log\left|\dfrac{4+x}{4-x}\right|+c$;

(*iii*) $\dfrac{1}{2\sqrt{7}}\log\left|\dfrac{\sqrt{7}+x}{\sqrt{7}-x}\right|+c$; (*iv*) $\dfrac{1}{2\sqrt{15}}\log\left|\dfrac{x-\sqrt{15}}{x+\sqrt{15}}\right|+c$;

2. (*i*) $\frac{1}{12}\log\left|\frac{2x-3}{2x+3}\right|+c$; (*ii*) $\frac{1}{12}\log\left|\frac{3+2x}{3-2x}\right|+c$;

(*iii*) $\frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{7}+x\sqrt{3}}{\sqrt{7}-x\sqrt{3}}\right|+c$; (*iv*) $\frac{1}{2\sqrt{105}}\log\left|\frac{x\sqrt{7}-\sqrt{15}}{x\sqrt{7}+\sqrt{15}}\right|+c$;

3. (*i*) $\frac{1}{10}\log\left|\frac{x-7}{x+3}\right|+c$; (*ii*) $\frac{1}{10}\log\left|\frac{3+x}{7-x}\right|+c$;

(*iii*) $\frac{1}{4}\log\left|\frac{x+1}{x+5}\right|+c$;

4. (*i*) $\frac{1}{\sqrt{17}}\log\left|\frac{4x+3-\sqrt{17}}{4x+3+\sqrt{17}}\right|+c$; (*ii*) $\frac{1}{\sqrt{17}}\log\left|\frac{\sqrt{17}+3+4x}{\sqrt{17}-3-4x}\right|+c$;

5. (*i*) $\frac{1}{2}\log|2x^2+3x-1|-\frac{3}{2\sqrt{17}}\log\left|\frac{4x+3-\sqrt{17}}{4x+3+\sqrt{17}}\right|+c$;

(*ii*) $\frac{5}{4}\log|1-3x-2x^2|-\frac{19}{4\sqrt{17}}\log\left|\frac{\sqrt{17}+3+4x}{\sqrt{17}-3-4x}\right|+c$;

5·6, Method of Substitution for some trigonometric expressions.

I. Integral of $\frac{1}{a+b\cos^2 x}$, $\frac{1}{a+b\sin^2 x}$.

If the integrand is $\frac{1}{a+b\cos^2 x}$, then divide the numerator and the denominator by $\cos^2 x$ and then proceed. If the integrand is $\frac{1}{a+b\sin^2 x}$, then divide the numerater and the denominater by $\sin^2 x$. The following examples will make this idea clear.

Illustrative Examples

Ex. 1. Evaluate the integrals

(*i*) $\int\frac{1}{3+4\cos^2 x}\,dx$; (*ii*) $\int\frac{1}{9+5\sin^2 x}\,dx$.

Sol. (*i*) $I=\int\frac{1}{3+4\cos^2 x}dx$

$$=\int\frac{\sec^2 x}{3\sec^2 x+4}dx$$

$$=\int\frac{\sec^2 x\,dx}{3(\tan^2 x+1)+4}\,dx$$

$$=\int\frac{\sec^2 x}{3\tan^2 x+7}\,dx.$$

Now, put $t=\tan x$.

Then $dt=\sec^2 x\,dx$.

Then we get,

$$I=\int \frac{1}{3t^2+7}\,dt$$

$$=\frac{1}{3}\int \frac{1}{t^2+\left(\sqrt{\frac{7}{3}}\right)}\,dx$$

$$=\frac{1}{3}\left(\frac{\sqrt{3}}{\sqrt{7}}\right)\tan^{-1}\left(\frac{t}{(\sqrt{7}/\sqrt{3})}\right)+c.$$

$$=\frac{1}{\sqrt{3}\sqrt{7}}\tan^{-1}\left(\frac{\sqrt{3}\,t}{\sqrt{7}}\right)+c.$$

$$=\frac{1}{\sqrt{21}}\left(\frac{\sqrt{3}\tan x}{\sqrt{7}}\right)+c.$$

(ii) Let $I=\int \frac{1}{9+5\sin^2 x}\,dx.$

Then $I=\int \frac{\operatorname{cosec}^2 x}{9\operatorname{cosec}^2 x+5}\,dx.$

$$=\int \frac{\operatorname{cosec}^2 x}{9\cot^2 x+14}\,dx.$$

Putting $t=\cot x$, we get

$dx=-\operatorname{cosec}^2 x\,dx.$

$$\therefore \quad I=-\int \frac{1}{9t^2+14}\,dt$$

$$=-\frac{1}{9}\int \frac{1}{t^2+\left(\frac{\sqrt{14}}{3}\right)^2}dt$$

$$=-\frac{1}{9}\cdot\frac{3}{\sqrt{14}}\tan^{-1}\frac{t}{\left(\frac{\sqrt{14}}{3}\right)}+c$$

$$=-\frac{1}{3\sqrt{14}}\tan^{-1}\frac{3t}{\sqrt{14}}+c.$$

$$=-\frac{1}{3\sqrt{14}}\tan^{-1}\left(\frac{3\cot x}{\sqrt{14}}\right)+c.$$

II. Integrals of $\left(\frac{\mathbf{a\cos x+b\sin x}}{\mathbf{c\sin x+d\cos x}}\right)$.

If we have to evaluate the integral of the type

$$\int \frac{a\cos x+b\sin x}{c\cos x+d\sin x}\,dx,$$

we may adopt the following procedure.

Put $(a \cos x+b \sin x)$

$$\equiv A(c \cos x+d \sin x).1 + B\left[\frac{d}{dx}(c \cos x+d \sin x)\right]$$

Express the numerater as

$$A(\text{denominator})+B\left[\frac{d}{dx}(\text{denominator})\right]$$

where A and B are the constants to be suitably determined. It is then easy to evaluate the integral.

Illustrative Example

Ex. 2. Evaluate $\int \frac{(2 \cos x+3 \sin x)}{(5 \cos x+7 \sin x)} dx$

Sol. Let

$$(2 \cos x+3 \sin x) \equiv A(5 \cos x+7 \sin x) + B\frac{d}{dx}\left(5 \cos x+7 \sin x\right).$$

$$\therefore \quad (2 \cos x+3 \sin x) \equiv A(5 \cos x+7 \sin x) + B(-5 \sin x+7 \cos x)$$

Equating the coefficients of sin x and cos x, we get

$$7A-5B=3 \quad \text{and}$$

$$5A+7B=2.$$

$$\therefore \quad A=\frac{217}{518}=\frac{31}{74}$$

and

$$B=-\frac{1}{74}$$

$$\therefore \quad (2 \cos x+3 \sin x)=\frac{31}{74}\left(5 \cos x+7 \sin x\right) - \frac{1}{74}\frac{d}{dx}\left(5 \cos x+7 \sin x\right)$$

$$\therefore \quad \int \frac{2 \cos x+3 \sin x}{5 \cos x+7 \sin x} dx$$

$$=\int \frac{\frac{31}{74}(5 \cos x+7 \sin x)-\frac{1}{74}\frac{d}{dx}(5 \cos x+7 \sin x)}{(5 \cos x+7 \sin x)} dx$$

$$=\frac{31}{74}\int 1 \cdot dx-\frac{1}{74}\int \frac{\frac{d}{dx}(5 \cos x+7 \sin x)}{(5 \cos x+7 \sin x)} dx$$

$$=\frac{31}{74}x-\frac{1}{74}\log\left|5 \cos x+7 \sin x\right|+c.$$

EXERCISE 5·6. (a)

Integrate the following functions :

1. (*i*) $\frac{1}{3-2\sin^2 x}$; (*ii*) $\frac{1}{2-3\cos^2 x}$;

(*iii*) $\frac{1}{4+3\sin^2 x}$; (*iv*) $\frac{1}{4+3\cos^2 x}$;

(*v*) $\frac{1}{9+16\cos^2 x}$; (*vi*) $\frac{1}{16+9\sin^2 x}$.

2. (*i*) $\frac{14\sin x+8\cos x}{2\sin x+3\cos x}$; (*ii*) $\frac{\cos x-\sin x}{\cos x+\sin x}$;

(*iii*) $\frac{7\sin x+6\cos x}{5\sin x-4\cos x}$ (*iv*) $\frac{3\sin x}{6\cos x+7\sin x}$;

(*v*) $\frac{1}{1+\cot x}$; (*vi*) $\frac{1}{1-\tan x}$;

(*vii*) $\frac{4\sin x-38\cos x}{5\sin x-11\cos x}$.

ANSWERS

1. (*i*) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x}{\sqrt{3}}\right)+c$;

(*ii*) $\frac{1}{\sqrt{8}}\log\left|\frac{\sqrt{2}\tan x-1}{\sqrt{2}\tan x+1}\right|+c$;

(*iii*) $-\frac{1}{2\sqrt{7}}\tan^{-1}\left(\frac{2\cot x}{\sqrt{7}}\right)+c$;

(*iv*) $\frac{1}{2\sqrt{7}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{7}}\right)+c$;

(*v*) $\frac{1}{15}\tan^{-1}\left(\frac{3\tan x}{5}\right)+c$;

(*vi*) $\frac{1}{20}\tan^{-1}\left(\frac{4\tan x}{5}\right)+c$.

2. (*i*) $4x-2\log|2\sin x+3\cos x|+c$;

(*ii*) $\log|\cos x+\sin x|+c$;

(*iii*) $\frac{11}{41}x-\frac{58}{41}\left|\log 5\sin x-4\cos x\right|+c$.

(*iv*) $\frac{21}{85}x-\frac{18}{85}\left|\log 6\cos x+7\sin x\right|+c$. ;

(*v*) $\frac{1}{2}x-\frac{1}{2}\log|\sin x+\cos x|+c$;

(*vi*) $\frac{1}{2}x-\frac{1}{2}\log|\cos x-\sin x|+c$;

(*vii*) $3x-\log|5\sin x-11\cos x|+c$.

III. Integrals of $\frac{1}{a+b\cos x}$, $\frac{1}{a+b\sin x}$.

In order to obtain the primitives of these functions, we use the substitution $\tan\frac{x}{2}=t$, so that,

$$\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2t}{1+t^2}, \quad \text{and}$$

$$\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}.$$

Also. $\frac{1}{2}\sec^2\frac{x}{2}\,dx=dt, \quad \therefore \quad \left(1+\tan^2\frac{x}{2}\right)dx=2\,dt.$

$\therefore \quad (1+t^2)\,dx=2dt.$

$\therefore \quad dx=\frac{2dt}{1+t^2}.$

Thus, by this substitution, the integrands $\frac{1}{a+b\cos x}$, $\frac{1}{a+b\sin x}$ get converted into algebraic form. The following examples will make the procedure clear.

Illustrative Examples

Ex. 1. Integrate the following :

(*i*) $\frac{1}{4+3\cos x}$; (*ii*) $\frac{1}{3-5\sin x}$;

(*iii*) $\frac{1}{2+\cos x-\sin x}$.

Sol. Put $t=\tan x/2$. Then we have,

$$dx=\frac{2}{1+t^2}\,dt, \quad \sin x=\frac{2t}{1+t^2}. \quad \text{and}$$

$$\cos x=\frac{1-t^2}{1+t^2}.$$

$$(i)\ \int\frac{1}{4+3\cos x}\,dx=\int\frac{1}{4+3\left(\frac{1-t^2}{1+t^2}\right)}\,\frac{2}{1+t^2}\,dt.$$

$$=\int\frac{2}{4(1+t^2)+3(1-t^2)}\,dt.$$

$$=\int\frac{2dt}{t^2+7}=\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{t}{\sqrt{7}}\right)+c.$$

$$=\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{1}{\sqrt{7}}\tan\frac{x}{2}\right)+c.$$

$$(ii)\ \int\frac{dx}{3-5\sin x}=\int\frac{2dt}{(1+t^2)\left(3-5\cdot\frac{2t}{1+t^2}\right)}=\int\frac{2dt}{3+3t^2-10t}$$

$$= \frac{2}{3}\int \frac{dt}{t^2 - \frac{10t}{3} + 1} = \frac{2}{3}\int \frac{dt}{t^2 - \frac{10t}{3} + \frac{25}{9} - \frac{16}{9}}$$

$$= \frac{2}{3}\int \frac{dt}{\left(t - \frac{5}{3}\right)^2 - \left(\frac{4}{3}\right)^2} = \frac{2}{3} \cdot \frac{3}{2 \cdot 4} \log \left| \frac{t - \frac{5}{3} - \frac{4}{3}}{t - \frac{5}{3} + \frac{4}{3}} \right| + c.$$

$$= \frac{1}{4} \log \left| \frac{3 \tan \frac{x}{2} - 9}{3 \tan \frac{x}{2} - 1} \right| + c.$$

(*iii*) $$\int \frac{dx}{2 + \cos x - \sin x} = \int \frac{\frac{2dt}{(1+t^2)}}{\left(2 + \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}\right)}$$

$$= \int \frac{2dt}{(2+2t^2) + (1-t^2) - 2t}$$

$$= \int \frac{2dt}{t^2 - 2t + 3} = 2\int \frac{dt}{(t-1)^2 + 2}$$

$$= \int \frac{2}{\sqrt{2}} \tan^{-1}\left(\frac{t-1}{\sqrt{2}}\right) + c.$$

$$= \sqrt{2} \tan^{-1}\left(\frac{\tan \frac{x}{2} - 1}{\sqrt{2}}\right) + c$$

EXERCISE 5·6 (b)

Integrate the following.

1. (*i*) $\frac{1}{3 + 2\cos x}$; (*ii*) $\frac{1}{4 + 5\cos x}$;

   (*iii*) $\frac{1}{5 - 4\cos x}$; (*iv*) $\frac{1}{4 - 5\cos x}$;

2. (*i*) $\frac{1}{5 + 4\sin x}$; (*ii*) $\frac{1}{4 + 5\sin x}$;

   (*iii*) $\frac{1}{5 - 4\sin x}$; (*iv*) $\frac{1}{1 - 3\sin x}$;

3. (*i*) $\frac{1}{2 + \sin x + \cos x}$; (*ii*) $\frac{1}{13 + 3\cos x + 4\sin x}$;

   (*iii*) $\frac{1}{2\sin x - \cos x + 3}$;

4. (*i*) $\frac{1}{3\cos 2x + 5}$; (*ii*) $\frac{1}{2 - 3\sin x}$;

ANSWERS

1. (*i*) $\frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{1}{\sqrt{5}} \tan \frac{x}{2}\right)+c$; (*ii*) $\frac{1}{3} \log\left(\frac{3+\tan \frac{x}{2}}{3-\tan \frac{x}{2}}\right)+c$

(*iii*) $\frac{2}{3} \tan^{-1}\left(3 \tan \frac{x}{2}\right)+c$; (*iv*) $\frac{1}{3} \log\left(\frac{3 \tan \frac{x}{2}-1}{3 \tan \frac{x}{2}+1}\right)+c$;

2. (*i*) $\frac{2}{3} \tan^{-1}\left(\frac{5 \tan \frac{x}{2}+4}{3}+c.\right)$

(*ii*) $\frac{1}{3} \log\left(\frac{1+2 \tan \frac{x}{2}}{4+2 \tan \frac{x}{2}}\right)+c$;

(*iii*) $\frac{2}{3} \tan^{-1}\left(\frac{5 \tan \frac{x}{2}-4}{3}\right)+c.$

(*iv*) $\frac{1}{2\sqrt{2}} \log\left(\frac{\tan \frac{x}{2}-3-2\sqrt{2}}{\tan \frac{x}{2}-3+2\sqrt{2}}\right)+c$;

3. (*i*) $\sqrt{2} \tan^{-1}\left(\frac{1+\tan \frac{x}{2}}{\sqrt{2}}\right)+c.$

(*ii*) $\frac{1}{6} \tan^{-1}\left(\frac{5 \tan \frac{x}{2}+2}{6}\right)+c$;

(*iii*) $\tan^{-1}\left(1+2\tan \frac{x}{2}\right)+c$;

4. (*i*) Hint : Put $\tan x=t$; $\frac{1}{4} \tan^{-1}\left(\frac{1}{2} \tan x\right)+c$;

(*ii*) $\frac{1}{2\sqrt{5}} \log\left|\left[\frac{2 \tan x-(3+\sqrt{5})}{2 \tan x-(3-\sqrt{5})}\right]\right|+c.$

5·7. Integration by parts.

Another method of finding integrals is integration by parts. This method can be used when the integrand is a product of two functions and one of them is a derivative of some known function or for one of the functions it is easy to find the integral.

The rule of integration is stated below

$\int f(x)g'(x)dx=\int f(x)g(x) \ -\int f'(x)\ g(x)\ dx.$

Proof. From the rule of derivative of a product of two functions, we have,

$$\frac{d}{dx}[f(x).\ g(x)]=f(x)\ g'(x)+f'(x)\ g(x).$$

$\therefore$ By the definition of integration, we have

$$f(x)\ g(x)=\int[f(x)\ g'(x)+f'(x)g(x)]\ dx$$
$$=\int f(x)\ g'(x)\ dx+\int f'(x)\ g(x)\ dx.$$

$$\therefore \int f(x)\ g'(x)\ dx=f(x)\ g(x)-\int f'(x)\ g(x)\ dx.$$

Remarks. In the above discussion, we have omitted the constant of integration.

2. If we put $f(x)=u$ and $g(x)=v$ then the above formula takes the form

$$\int u\left(\frac{dv}{dx}\right)dx=uv-\int v\left(\frac{du}{dx}\right)dx. \qquad ..(1)$$

Illustrative Examples.

Ex. 1. Evaluate the integrals of the following functions $x\cos x$;

Sol. We shall use (1).

Put $u=x$ and $\frac{dv}{dx}=\cos x.$

Then $\frac{du}{dx}=1$ and $v=\sin x.$

$$\therefore \int x\cos x\ dx.=\sin x-\int \sin x.\ 1.\ dx.$$
$$=x\sin x-(-\cos x)+c$$

$$\therefore \int x\cos x\ dx=x\sin x+\cos x+c.$$

Remark. In the integral $\int x\cos x\ dx$ it is easy to find the integrals of the both x and $\cos x$ where product is the integrand $x\cos x$. Naturally, any function can be chosen as u and the other one $\frac{dv}{dx}$. Let us now see what happens if we choose $u=\cos x$ and $\frac{dv}{dx}=x.$

Let $u=\cos x$ and $\left(\frac{dv}{dx}\right)=x.$

Then clearly, $\frac{du}{dx}=-\sin x$ and $v=\frac{x^2}{2}.$

$\therefore$ (1) gives us.

$$\int(\cos x)\ x\ dx=(\cos x)\left(\frac{x^2}{2}\right)-\int\frac{x^2}{2}.(-\sin x)\ dx+c.$$

$$= \frac{x^2}{2}\cos x + \frac{1}{2}\int x^2 \sin x \, dx.$$

Upto this stage, the things have gone well but now we shall have to evaluate the integral

$$\int x^2 \sin x \, dx.$$

which is more *complicated* that the integral

$$\int x \cos x \, dx.$$

It is now evident that it is not enough to know the formula of integration by parts but one should also know how to choose the functions u and $\frac{dv}{dx}$ in the product.

In the present course, we are studying algebraic (A), trigonometric (T), exponential (E), logarithmic (L) and inverse trigonometric (I) functions. It is advisable to remember the rule **LIATE**. In the product, whenever we want to choose u, we choose the one whose alphabet occurs first in the order **LIATE**.

The following examples will make the procedure further clear.

Ex. 2. Evaluate (*i*) $\int x \log x \, dx$; (*ii*) $\int x \tan^{-1} x \, dx$; (*iii*) $\int x^2 e^x$.

Sol. (1) $I = \int x \log x \, dx$.

Here we choose $u = \log x$ and $\frac{dv}{dx} = x$ as per the **LIATE** scheme.

Clearly, $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^2}{2}$.

$\therefore$ Applying the formula (**1**), we get.

$$I = \int x \log x \, dx = (\log x)\left(\frac{x^2}{2}\right) - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx.$$

$$= \frac{x^2 \log x}{2} - \frac{1}{2}\int x \, dx.$$

$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + c.$$

(*ii*) Let $I = \int x \tan^{-1} x \, dx$.

Here, we choose $u = \tan^{-1} x$ and $= \frac{dv}{dx} = x$ as per **LIATE** scheme.

Clearly, we have $\frac{du}{dx} = \frac{1}{1+x^2}$ and $v = \frac{x^2}{2}$.

$\therefore$ Applying formula (**1**), we get.

$$I = \int x \tan^{-1} x \, dx = (\tan^{-1} x)\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} \, dx.$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1-1}{1+x^2}\, dx.$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2}\left[\int 1.\, dx - \int \frac{1}{1+x^2}\, dx.\right]$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{\tan^{-1} x}{2} + c.$$

$$= \frac{1}{2}(x^2+1) \tan^{-1} x - \frac{x}{2} + c.$$

(*iii*) Let $I = \int x^2 e^x\, dx$.

Choose $u = x^2$ and $\frac{dv}{dx} = e^x$,

$\therefore$ $\frac{du}{dx} = 2x$ and $v = e^x$

$\therefore$ Formula (1) gives us,

$$\int x^2 e^x\, dx = x^2 e^x - \int e^x.\, 2x\, dx.$$

$$= x^2 e^x - 2\int x e^x\, dx.$$

Now, let $I_1 = \int x e^x\, dx$.

Applying formula (1) of integration by parts, we get

$$I_1 = x e^x - \int e^x.\, 1.\, dx$$

$$= x e^x - e^x.$$

$\therefore$ $\int x^2 e^x\, dx = x^2 e^x - 2\,[x e^x - e^x] + c.$

$$= (x^2 - 2x + 2)\, e^x + c.$$

In this case, we have applied the method of integration by parts twice to evaluate the integral.

Ex. 3. Evaluate (*i*) $\int \log x\, dx$, (*ii*) $\int \tan^{-1} x\, dx$.

Sol. In both the examples above, the integrands do not seem to be in the form of product of two functions. However they can be certainly expressed as product of functions by taking the second function as 1.

(*i*) $\int \log x\, dx = \int 1.\, \log x\, dx$.

Put $u = \log x$ and $\frac{dv}{dx} = 1$.

$\therefore$ $\frac{du}{dx} = \frac{1}{x}$ and $v = x$.

$\therefore$ Applying the formula (1) of integration by parts, we get

$$\int \log x\, dx = (\log x).\, x - \int x.\, \frac{1}{x}\, dx.$$

$$= x \log x - \int 1.\, dx.$$

$$= x \log x - x + c.$$

(*ii*) $\int \tan^x x\, dx = \int 1.\ \tan^{-1} x\, dx.$

Put $u = \tan^{-1} x$ and $\frac{dv}{dx} = 1.$

Clearly we have $\frac{du}{dx} = \frac{1}{1+x^2}$ and $v = x.$

$\therefore$ On application of (1) of integration by parts, we get

$$\int \tan^{-1} x\, dx = (\tan^{-1} x)\ x - \int x.\ \frac{1}{1+x^2}\ dx.$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\ dx.$$

$$= x\tan^{-1} x - \frac{1}{2}\ \log(1+x^2) + c.$$

Ex. 4. Evaluate (*i*) $\int x^2 \cos x\, dx$; (*ii*) $\int x^2\ e^{3x}\, dx.$
(*iii*) $\int x^5 \log x\, dx.$

Sol. (*i*) For $\int x^2 \cos x\, dx$, choose $u = x^2, \frac{dv}{dx} = \cos x.$

$\therefore$ $\frac{du}{dx} = 2x$ and $v = \sin x.$

$\therefore$ Applying formula (1) of integration by parts, we get,

$$\int x^2 \cos x\, dx = x^2\ (\sin x) - \int (\sin x)\ 2x\, dx.$$

$$= x^2 \sin x - 2\int x \sin x\, dx.$$

Again applying integration by parts, we get

$$\int x \sin x = x\ (-\cos x) - \int (-\cos x).\ 1.\ dx$$

$$= -x \cos x + \sin x.$$

$\therefore$ $\int x^2 \cos x\, dx = x^2 \sin x - 2\ [-x \cos x + \sin x] + c.$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c.$$

(*ii*) For $\int x^2\ e^{3x}\ dx$, choose

$$u = x^2 \text{ and } \frac{dv}{dx} = e^{3x}$$

$\therefore$ $\frac{du}{dx} = 2x$ and $v = \frac{1}{3}\ e^{3x}.$

$\therefore$ Applying formula (1) of integration by parts, we get

$$\int x^2\ e^{3x}\ dx = x^2 \left(\frac{e^{3x}}{3}\right) - \int \frac{e^{3x}}{3}.\ 2x\, dx$$

$$= \frac{x^2\ e^{3x}}{3} - \frac{2}{3}\int x\ e^{3x}\, dx.$$

Once again, by method of integration by parts, we get

$$\int x\, e^{3x}\, dx = x\left(\frac{e^{3x}}{3}\right) - \int\left(\frac{e^{3x}}{3}\right).\ 1.\ dx$$

$$= \frac{xe^{3x}}{3} - \frac{1}{9}.\ e^{3x}.$$

$\therefore$ We get,

$$\int x^2\, e^{3x}\, dx = \frac{x^2\, e^{3x}}{3} - \frac{2}{3}\left[\frac{xe^{3x}}{3} - \frac{1}{9}\, e^{3x}\right] + c.$$

$$= \frac{1}{3}\, x^2\, e^{3x} - \frac{2}{9}\, xe^{3x} + \frac{2}{27}\, e^{3x} + c.$$

$$= e^{3x}\left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right) + c.$$

(iii) For $\int x^5 \log x\, dx$, we choose $u = \log x$ and $\frac{dv}{dx} = x^5$

Then, we get $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^6}{6}$

$\therefore$ Applying formula (1) of integration by parts, we get

$$\int x^5 \log x\, dx = (\log x)\, \frac{x^6}{6} - \int \frac{x^6}{6}\ .\ \frac{1}{x}\, dx.$$

$$= \frac{x^6 \log x}{6} - \frac{1}{6}\int x^5\, dx.$$

$$= \frac{x^6 \log x}{6} - \frac{1}{6}\cdot\left(\frac{x^6}{6}\right) + c.$$

$$= \frac{x^6 \log x}{6} - \frac{x^6}{36} + c.$$

$$= \left(\log x - \frac{1}{6}\right)\frac{x^6}{6} + c.$$

Ex. 5. Evaluate $\int e^{ax} \cos(bx+c)\, dx$.

Sol. Let $I = \int e^{ax} \cos(bx+c)\, dx$.

Choose $u = \cos(bx+c)$ and $\frac{dv}{dx} = e^{ax}$

Then, $\frac{du}{dx} = -b \sin(bx+c)$ and $v = \frac{1}{a}\, e^{ax}$

$\therefore$ Applying formula (1) of integration by parts, we get

$$I = \int e^{ax} \cos(bx+c)\, dx = \cos(bx+c).\ \left(\frac{e^{ax}}{a}\right)$$

$$- \int \frac{e^{aw}}{a}\ [-b \sin(bx+c)]\, dx.$$

$$=\frac{1}{a}\ e^{ax}\cos\ (bx+c)+\frac{b}{a}\int e^{ax}\sin\ (bx+c)\ dx.$$

Now applying method of integration by parts, we get

$$\int e^{ax}\sin\ (bx+c)\ dx=\sin\ (bx+c).\ \frac{e^{ax}}{a}$$

$$-\int\left(\frac{e^{ax}}{a}\right).\ b\cos\ (bx+c)\ dx.$$

$$=\frac{1}{a}\ e^{ax}\sin\ (bx+c)-\frac{b}{a}\int e^{ax}\cos\ (bx+c)\ dx.$$

$$=\frac{1}{a}\ e^{ax}\sin\ (bx+c)-\frac{b}{a}\ I.$$

Substituting this value in I, we get

$$I=\frac{1}{a}\ e^{ax}\cos\ (bx+c)+\frac{b}{a}\left[\frac{1}{a}\ e^{ax}\sin\ (bx+c)-\frac{b}{a}\ I.\right]$$

$$\therefore\ \left(1+\frac{b^2}{a^2}\right)I=\frac{e^{ax}}{a^2}\ [a\cos\ (bx+c)+b\sin\ (bx+c)]$$

$$\therefore\ \ I=\frac{e^{ax}}{a^2+b^2}\ [a\cos\ (bx+c)+b\sin\ (bx+c)]$$

$$\therefore\ \int e^{ax}\cos\ (bx+c)\ dx.$$

$$=\frac{e^{ax}}{a^2+b^2}\ [a\cos\ (bx+c)+b\sin\ (bx+c)]+k,$$

k is a constant of integration.

EXERCISE 5·7

Integrate the following :

1. (*i*) $x\cos x$; (*ii*) $x\sec^2 x$; (*iii*) $x^3\sin x$; (*iv*) $x\sin x\cos x$; (*v*) $x\tan^2 x$.

2. (*i*) xe^{2x} ; (*ii*) x^2a^x ; (*iii*) x^3e^{-x} ; (*iv*) $x^3e^{x^2}$; (*v*) $e^{\sin x}\sin 2x$.

3. (*i*) $x^2\log x$; (*ii*) $x^2\tan^{-1}x$; (*iii*) $x(\tan^{-1}x)^2$; (*iv*) $x(\log x)^2$; (*v*) $x\log(x+1)$.

4. (*i*) $\sin^{-1}x$; (*ii*) $(\log x)^2$;

   (*iii*) $\dfrac{\log(\log x)}{x}$; (*iv*) $\cos x\log(\sin x)$

5. (*i*) $\cos\sqrt{x}$; (*ii*) $\sin\sqrt{x}$; (*ii*) $\tan^{-1}\sqrt{x}$;

6. (*i*) $\dfrac{x}{1-\cos x}$; (*ii*) $\dfrac{x}{1+\cos x}$;

   (*iii*) $\dfrac{x}{1+\sin x}$; (*iv*) $\dfrac{x+\sin x}{1-\cos x}$

7. (*i*) $e^{5x}\cos 7x$; (*ii*) $e^{2x}\sin x\cos x$; (*iii*) $e^{4x}\sin(3x-1)$; (*iv*) $2^x\cos 3x$; (*v*) $a^{2x+1}\sin(2x+1)$.

8. (*i*) $(x+3)\tan^{-1}(x+3)$; (*ii*) $x^5\tan^{-1}x^3$; (*iii*) $x\tan x\sec^2 x$.

9. $e^{2x}\sin(e^x+3)$;

10. (*i*) $\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}$; (*ii*) $\dfrac{x\tan^{-1}x}{(1+x^2)^{3/2}}$

(*iii*) $\dfrac{(x-1)^2\tan^{-1}(x-1)}{x^2-2x+2}$.

ANSWERS

1. (*i*) $x\sin x+\cos x+c$; (*ii*) $x\tan x-\log|\sec x|+c$;

(*iii*) $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+c$;

(*iv*) $-\dfrac{x}{4}\cos 2x+\dfrac{1}{8}\sin 2x$; (*v*) **Hint :** Use $\tan^2 x=\sec^2 x-1$;

Ans. $x\tan x-\log|\sec x-|-\dfrac{x^2}{2}+c$.

2. (*i*) $\dfrac{e^{2x}}{2}\left(x-\dfrac{1}{2}\right)+c$; (*ii*) $\dfrac{a^x}{(\log a)^3}[x^2(\log a)^2-2x\log a+2]+c$.

(*iii*) $-e^{-x}(x^3+3x^2+6x+6)+c$. (*iv*) **Hint :** Put $x^2=t$;

Ans . $\dfrac{e^{-x2}}{2}(x^2-1)+c$; (*v*) **Hint :** Put $\sin x=t$.

Ans : $2e^{\sin x}(\sin x-1)+c$.

3. (*i*) $\dfrac{x^3}{9}(3\log x-1)+c$; (*ii*) $\dfrac{x^3}{3}\tan^{-1}x-\dfrac{x^2}{6}+\dfrac{1}{6}\log(x^2+1)+c$;

(*iii*) $\frac{1}{2}(x^2+1)(\tan^{-1}x)^2-x\tan^{-1}x+\frac{1}{2}\log(1+x^2)+c$;

(*iv*) $\dfrac{x^2}{2}(\log x)^2-\dfrac{x^2}{2}\log x+\dfrac{x^2}{4}+c$.

(*v*) $\dfrac{x^2-1}{2}\log(x+1)-\dfrac{x^2}{4}+\dfrac{x}{2}+c$.

4. (*i*) $x\sin^{-1}x+\sqrt{1-x^2}+c$; (*ii*) $x(\log x)^2-2x\log x+2x+c$;

(*iii*) $\log x[\log(\log x)-1]+c$; (*iv*) **Hint :** Put $\sin x=t$.

Ans. $\sin x(\log\sin x-1)+c$.

5. (*i*) **Hint :** Put $x=t^2$ **Ans :** $2(\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x})+c$;

(*ii*) $2(-\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x})+c$;

(*iii*) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+c$.

6. (*i*) $-x(\cot x+\operatorname{cosec}x)+\log|1-\cos x|+c$; *i.e.*

$$-x\cot\frac{x}{2}+2\log\left|\sin\frac{x}{2}\right|+c;$$

(*ii*) $x\tan\dfrac{x}{2}+2\log\left|\cos\dfrac{x}{2}\right|+c$;

(*iii*) **Hint :** $\dfrac{x}{1+\sin x}=\dfrac{x(1-\sin x)}{\cos^2 x}=x(\sec^2 x-\sec x\tan x)$.

Ans : $x(\tan x-\sec x)+\log|1+\sin x|+c$;

(*iv*) $-x\cot\left(\dfrac{x}{2}\right)+4\log\left|\sin\dfrac{x}{2}\right|+c$.

7. (*i*) $\frac{e^{5x}}{74}(5\cos 7x+7\sin 7x)+c$:

(*ii*) $\frac{e^{2x}}{8}(\sin 2x-\cos 2x)+c$;

(*iii*) $\frac{e^{4x}}{25}[4\sin(3x-1)-3\cos(3x-1)]+c$;

(*iv*) $\frac{2^x}{9+(\log 2)^2}[(\log 2)\cos 3x+3\sin 3x]+c$.

(*v*) Hint : Put $2x+1=t$;

Ans : $\frac{a^{2x+1}}{2[1+(\log a)^2]}[(\log a)\cdot\sin(2x+1)-\cos(2x+1)]+c$

8. (*i*) $\frac{1}{2}(x^2+6x+10)\tan^{-1}(x+3)-\frac{x}{2}+c$;

(*ii*) **Hint** : Put $x^3=t$;

Ans. $\frac{1}{6}[(1+x^6)\tan^{-1}x^3-x^3]+c$;

(*iii*) **Hint** : $\int\tan x\sec^2 x\,dx=\frac{1}{2}\tan^2 x$

Ans : $\frac{x}{2}\tan^2 x-\frac{1}{2}\tan x+\frac{x}{2}+c$;

9. Hint : Put $e^x=t$;

Ans : $\sin(e^x+3)-e^x\cdot\cos(e^x+3)+c$.

10. (*i*) **Hint** : Put $\sin^{-1}x=t$;

Ans. : $x-\sqrt{1-x^2}\sin^{-1}x+c$;

(*i*) $\frac{1}{\sqrt{1+x^2}}(x-\tan^{-1}x)+c$;

(*ii*) $(x-1)\tan^{-1}(x-1)-\frac{1}{2}\log|x^2-2x+2|-\frac{1}{2}[\tan^{-1}(x-1)]^2+c$.

5.8. Application of Method of Substitution

I. Integrals of $\sqrt{x^2+a^2}$; $\sqrt{a^2-x^2}$; $\sqrt{x^2-a^2}$.

In this section, we shall be making use of the formulae for the integrals of $\frac{1}{\sqrt{a^2-x^2}}$: $\frac{1}{\sqrt{x^2+a^2}}$, $\frac{1}{\sqrt{x^2-a^2}}$ obtained earlier.

(*i*) Let $I=\int\sqrt{a^2-x^2}\,dx$.

Integrating by parts, we get

$$I=\int 1.\sqrt{a^2-x^2}\,dx$$

$$=\sqrt{a^2-x^2}\cdot x-\int\frac{(-2x)}{2\sqrt{a^2-x^2}}.x\,dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{(-x^2)}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}dx$$

$$= x\sqrt{a^2-x^2} - \left[\int \sqrt{a^2-x^2}\,dx - a^2\int \frac{1}{\sqrt{a^2-x^2}}\,dx\right]$$

$$= x\sqrt{a^2-x^2} - I + a^2 \sin^{-1}\left(\frac{x}{a}\right).$$

$$\therefore \quad 2I = x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right).$$

$$\therefore \quad I = \frac{1}{2}\left[x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right].$$

$$\therefore \int \sqrt{a^2-x^2}\,dx = \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] + c.$$

(*ii*) Let $I = \int \sqrt{x^2-a^2}\,dx.$

Integrating by parts, we get

$$I = \int 1.\sqrt{x^2-a^2}\,dx$$

$$= \sqrt{x^2-a^2}.x - \int x.\frac{2x}{2\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{a^2-x^2}\,dx - a^2\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - I - a^2 \log\left|x + \sqrt{x^2-a^2}\right|.$$

$$\therefore \quad 2I = x\sqrt{x^2-a^2} - a^2 \log\left|a + \sqrt{x^2-a^2}\right|.$$

$$\therefore \quad I = \tfrac{1}{2}\left[x\sqrt{x^2-a^2} - a^2 \log\left|x + \sqrt{x^2-a^2}\right|\right].$$

$$\therefore \int \sqrt{x^2-a^2}\,dx = \tfrac{1}{2}\left[x\sqrt{x^2-a^2} - a^2 \log\left|x + \sqrt{x^2-a^2}\right|\right] + c.$$

(*ii*) Let $I = \int \sqrt{x^2+a^2}\,dx.$

Integrating by parts, we get

$$I = \int 1.\sqrt{x^2+a^2}\,dx$$

$$= \sqrt{x^2+a^2}.x - \int x.\,2\frac{2x}{\sqrt{x^2+a^2}}\,dx$$

$$= x\sqrt{x^2+a^2}dx - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$$

$$I = x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,dx + a^2 \int \frac{1}{\sqrt{x^2+a^2}}\,dx$$

$$= x\sqrt{x^2+a^2} - I + a^2 \log |x + \sqrt{x^2+a^2}|.$$

$$\therefore \quad 2I = x\sqrt{x^2+a^2} + a^2 \log |x + \sqrt{x^2+a^2}|.$$

$$\therefore \quad I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2+a^2} \right|.$$

$$\therefore \int \sqrt{x^2+a^2}\,dx = \tfrac{1}{2}[\sqrt{x^2+a^2} + a^2 \log |x + \sqrt{x^2+a^2}|] + c.$$

We now add these results to our list of standard formulae.

36. $$\int \sqrt{a^2-x^2}\,dx = \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] + c.$$

37. $$\int \sqrt{x^2-a^2}\,dx = \left[\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left| x + \sqrt{x^2-a^2}\right|\right] + c.$$

38. $$\int \sqrt{x^2+a^2}\,dx = \left[\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left| x + \sqrt{x^2+a^2}\right|\right] + c.$$

Illustrative Examples.

Ex. 1. Evaluate (*i*) $\int \sqrt{5-3x^2}\,dx$; (*ii*) $\int \sqrt{x^4+a^4}\,dx$

Sol. $$\int \sqrt{5-3x^2}\,dx = \sqrt{3}\int \sqrt{\frac{5}{3} - x^2}\,dx$$

$$= \sqrt{3}\left[\frac{x}{2}\sqrt{\frac{5}{3}-x^2} + \frac{(5/3)}{2}\cdot \sin^{-1}\left(\frac{x}{\sqrt{5/3}}\right)\right] + c.$$

$$= \sqrt{3}\left[\frac{x}{2}\sqrt{\frac{5}{3}-x^2} + \frac{5}{6}\sin^{-1}\left(\frac{\sqrt{3}\,x}{\sqrt{5}}\right)\right] + c.$$

(*ii*) $\int x\sqrt{x^4+a^4}\,dx = \frac{1}{2}\int \sqrt{(x^2)^2+(a^2)^2}\,2x\,dx.$

Put $x^2 = t.$

$\therefore \quad 2x\,dx = dt.$

$$\therefore \int x\sqrt{x^4+a^4}\,dx = \tfrac{1}{2}\int \sqrt{t^2+(a^2)^2}\,dx$$

$$= \frac{1}{2}\left[\frac{t}{2}\sqrt{t^2+a^4} + \frac{a^4}{2}\log\left|t + \sqrt{t^2+a^4}\right|\right] + c.$$

$$= \frac{x^2}{4}\sqrt{x^4+a^4} + \frac{a^4}{4}\log\left| x^2 + \sqrt{x^4+a^4}\right|\right] + c.$$

Ex. 2. Evaluate $\int \sqrt{3x^2-4x+7}\,dx.$

Sol. $$\int \sqrt{3x^2-4x+7}\,dx = \sqrt{3}\int \sqrt{x^2 - \frac{4}{3}x + \frac{7}{3}}\,dx$$

$$= \sqrt{3}\int \sqrt{\left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) + \frac{17}{9}}\,dx$$

$$=\sqrt{3}\int\sqrt{\left(x-\frac{2}{3}\right)+\left(\frac{\sqrt{17}}{2}\right)^2}\,dx$$

$$=\sqrt{3}\left[\frac{\left(x-\frac{2}{3}\right)}{2}\sqrt{x^2-\frac{4}{3}x+\frac{7}{3}}+\frac{\left(\frac{17}{9}\right)}{2}\log\left|\left(x-\frac{2}{3}\right)+\sqrt{x^2-\frac{4}{3}x+\frac{7}{3}}\right|\right]+c.$$

$$=\frac{3x-2}{6}\sqrt{3x^2-4x+7}+\frac{17\sqrt{3}}{18}\log\left|\left(x-\frac{2}{3}\right)+\sqrt{x^2-\frac{4}{3}x+\frac{7}{3}}\right|+c.$$

II. Integrals of $(px+q)\sqrt{ax^2+bx+c}$

We know that

$$\frac{d}{dx}(ax^2+bx+c)=2ax+b.$$

Therefore, we put

$$px+q\equiv A(2ax+b)+B.$$

This gives us

$$\int(px+q)\sqrt{ax^2+bx+c}\,dx$$

$$=\int A(2ax+b)\sqrt{ax^2+bx+c}\,dx+B\int\sqrt{ax^2+bx+c}\,dx$$

Now, $\int A(2ax+b)\sqrt{ax^2+bx+c}\,dx$ can be evaluated by substituting $ax^2+bx+c=t$; where as $\int\sqrt{ax^2+bx+c}\,dx$ can be evaluated using the above mentioned standard formulae.

Ex. 3. Evaluate $\int(x+3)\sqrt{3-4x-x^2}\,dx$

Sol. We have,

$$\frac{d}{dx}(3-4x-x^2)=-4-2x.$$

Now, let $x+3\equiv A(-4-2x)+B.$

Equating coefficients of x and the constant terms, we get

$$-2A=1$$

and $$-4A+B=3,$$

$\therefore$ $A=-\frac{1}{2}$ and $B=1.$

$\therefore$ $x+3=-\frac{1}{2}(-4-2x)+1.$

$\therefore$ $\int(x+3)\sqrt{3-4x-x^2}\,dx$

$$=-\tfrac{1}{2}\int(-4-2x)\sqrt{3-4x-x^2}\,dx+\int\sqrt{3-4x-x^2}\,dx.$$

Now, if we substitute

$$t = 3-4x-x^2 \text{ , then we get}$$
$$dt = (-4-2x)\,dx.$$

This gives,

$$\int(-4-2x)\sqrt{3-4x-x^2}\,dx$$
$$= \int \sqrt{t}\,dt = 2/3\; t^{3/2}$$
$$= \tfrac{2}{3}(3-4x-x^2)^{3/2},$$

Also, $\int \sqrt{3-4x-x^2}\,dx$

$$= \int \sqrt{7-4-4x-x^2}\,dx$$
$$= \sqrt{(\sqrt{7})^2-(x+2)^2}\,dx$$
$$= \frac{x+2}{2}\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right).$$

Putting these values in the given integral, we get

$$\int(x+3)\sqrt{3-4x-x^2}\,dx$$
$$= \left(\frac{-1}{2}\right)\frac{2}{3}(3-4x-x^2)^{3/2}$$
$$+\left[\frac{(x+2)}{2}\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right)\right]+c$$
$$= -\tfrac{1}{3}(3-4x-x^2)^{3/2}$$
$$+\left[\frac{(x+2)}{2}\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right)\right]+c.$$

EXERCISE 5·8.

Integrate with respect to x the following.

1. (*i*) $\sqrt{9x^2-16}$; (*ii*) $\sqrt{9x^2+16}$

(*iii*) $\sqrt{16-9x^2}$.

2. (*i*) $\sqrt{7x^2+5}$; (*ii*) $\sqrt{5-7x^2}$;

(*iii*) $\sqrt{7x^2-5}$.

3. (*i*) $\sqrt{8+2x-x^2}$; (*ii*) $\sqrt{1-x+x^2}$;

(*iii*) $\sqrt{x^2-4x}+3$; (*iv*) $\sqrt{2x-x^2}$.

4. (*i*) $\sqrt{4x^2-4x+3}$; (*ii*) $\sqrt{4+3x-2x^2}$;

(*iii*) $\sqrt{10-4x+4x^2}$.

5. (*i*) $(x+2)\sqrt{x^2-3x+7}$; (*ii*) $(x+2)\sqrt{x^2-x+1}$;

(*iii*) $(2x+4)\sqrt{2x^2+3x-1}$.

6. (*i*) $x\sqrt{4x-x^2+1}$; (*ii*) $x^2\sqrt{x^6-a^6}$.

ANSWERS

1. (*i*) $\frac{x}{2}\sqrt{9x^2-16}-\frac{8}{3}\log\left|3x+\sqrt{9x^2-16}\right|+c$;

(*ii*) $\frac{x}{2}\sqrt{9x^2-16}+\frac{8}{3}\log\left|3x+\sqrt{9x^2+16}\right|+c$;

(*iii*) $\frac{x}{2}\sqrt{16-9x^2}+\frac{8}{3}\sin^{-1}\left(\frac{3x}{4}\right)+c$.

2. (*i*) $\frac{x}{2}\sqrt{7x^2+5}+\frac{5}{2\sqrt{7}}\log\left|x\sqrt{7}+\sqrt{7x^2+5}\right|+c$;

(*ii*) $\frac{x}{2}\sqrt{5-7x^2}+\frac{5}{2\sqrt{7}}\sin^{-1}\left(\frac{x\sqrt{7}}{\sqrt{5}}\right)+c$;

(*iii*) $\frac{x}{2}\sqrt{7x^2-5}-\frac{5}{2\sqrt{7}}\log\left|x\sqrt{7}+\sqrt{7x^2-5}\right|+c$.

3. (*i*) $\left(\frac{x-1}{2}\right)\sqrt{8+2x-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x-1}{3}\right)+c$;

(*ii*) $\left(\frac{2x-1}{4}\right)\left|\sqrt{1-x+x^2}+\frac{3}{8}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{1-x+x^2}\right|\right.$

$+c$;

(*iii*) $\left(\frac{x-2}{2}\right)\sqrt{x^2-4x+3}-\frac{1}{2}\log\left|(x-2)+\sqrt{x^2-4x+3}\right|+c$;

(*iv*) $\left(\frac{x-1}{2}\right)\sqrt{2x-x^2}+\frac{1}{2}\sin^{-1}(x-1)+c$.

4. (*i*) $\left(\frac{2x-1}{4}\right)\sqrt{4x^2-4x+3}+\frac{1}{2}\log\left|(2x-1)+\sqrt{4x^2-4x+3}\right|c$.

(*ii*) $\left(\frac{4x-3}{8}\right)\sqrt{4+3x-2x^2}+\frac{41}{15\sqrt{2}}\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)+c$.

(*iii*) $\left(\frac{2x-1}{4}\right)\sqrt{10-4x+4x^2}+\frac{9}{4}\log\left|2x-1)+\right.$

$\left.\sqrt{10-4x-4x^2}\right|+c$.

5. (*i*) $\frac{1}{3}(x^2-2x+7)^{3/2}+\frac{7}{8}(2x-3)\sqrt{x^2-3x+7}$

$+\frac{133}{16}\log\left|(2x-3)+2\sqrt{x^2-3x+7}\right|+c$.

(*ii*) $\frac{1}{3}(x^2-x+1)^{3/2}+\frac{5}{8}(2x-1)\sqrt{x^2-x+1}$

$+\frac{15}{16}\log\left|(2x-1)+2\sqrt{x^2-x+1}\right|+c$.

(*iii*) $\frac{1}{3}(2x^2+3x-1)^{3/2}+\frac{5}{16}(4x+3)\sqrt{2x^2+3x-1}$

$-\frac{85}{32\sqrt{2}}\log\left|(4x+3)+\sqrt{2x^2+3x-1}\right|+c$.

6. (*i*) **Hint :** Put $x^2=t$; **Ans.** $\frac{2x^2-1}{8}\sqrt{x^4-x^2+1}$

$+\frac{3}{8}\log\left|\left(x^2-\frac{1}{2}\right)+\sqrt{x^4-x^2+1}\right|+c$.

(*ii*) $\frac{x^3}{6}\sqrt{x^6-a^6}-\frac{a^6}{6}\log\left|x^3+\sqrt{x^6-a^6}\right|+c$.

5·9. Reduction formulae for $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$.

In the earlier discussion, we have seen that it is quite simple to evaluate integrals of $\sin x$, $\sin^2 x$, $\cos x$, $\cos^2 x$ etc. However, to evaluate the integrals of higher powers $\sin x$ and $\cos x$ may seem to be, rather difficult. But, fortunately it is not so. There is a standard method—called as *reduction formula*—by means of which integrals of any higher integral powers of $\sin x$ and $\cos x$ can be evaluated.

(i) Reduction formula for $\int \sin^n x\,dx$

Let $I_n = \int \sin^n x\,dx$, where n is a positive integer greater than 2.

$$\therefore \qquad I = \int \sin^{(n-1)} x \sin x\,dx.$$

Here, we put $x = \sin^{n-1} u$ and $\frac{dv}{dx} = \sin x$.

Clearly, $\frac{du}{dx} = (n-1)\sin^{n-2} x \cos x$

and $v = (-\cos x)$.

$\therefore$ Applying formula (1) of integration by parts, we get

$$I_n = (\sin^{n-1} x)\ (-\cos x) - \int (-\cos x)(n-1)\sin^{n-2} x \cos x\,dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

$$\therefore \quad I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)\ I_n.$$

$$\therefore \quad nI_n = -\cos x \sin^{n-1} x + (n-1)\ I_{n-2}.$$

$$\therefore \quad I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \left(\frac{n-1}{n}\right) I_{n-2}. \qquad \ldots 1$$

By this formula, we can express the integral of $\sin^n x$ in terms of integral of $\sin^{n-2} x$ or in terms of reduced power of $\sin x$; and hence it is called a *reduction formula.*

Replacing n by $(n-2)$ in (1) we get

$$I_{n-2} = -\frac{1}{n-2}\cos x \sin^{n-3} x + \frac{n-3}{n-2} I_{n-4}.$$

Similarly, we get

$$I_{n-4}=-\frac{1}{n-4}\cos x\sin^{n-5}x+\frac{n-5}{n-4}I_{n-6}$$

and so on.

Continuing this way, we can get finally a relation which connects I_n with I_1 or I_0.

Now,

$$I_1=\int\sin^1 x\,dx=\int\sin x\,dx=-\cos x$$

and $$I_0=\int 1.dx=x.$$

Substituting all these values in (1), we can evaluate the integral $\sin^n x$ completely.

Illustrative Example

Ex. 1. Evaluate $\int\sin^6 x\,dx$ with help of the reduction formula.

Sol. We know that

$$I_n=-\frac{1}{n}\sin^{n-1}x\cos x+\frac{n-1}{n}I_{n-2}$$

Putting $n=2$, 4 and 6 in succesion, we get

$$I_2=-\tfrac{1}{2}\sin x\cos x+\tfrac{1}{2}I_0$$
$$=-\tfrac{1}{2}\sin x\cos x+\tfrac{1}{2}.x\ ;$$
$$I_4=-\tfrac{1}{4}\sin^3\cos x+\tfrac{3}{4}I_2$$
$$=-\tfrac{1}{4}\sin^3 x\cos x+\tfrac{3}{4}[-\tfrac{1}{2}\sin x\cos x+\tfrac{1}{2}x]$$
$$=-\tfrac{1}{4}\sin^3 x\cos x-\tfrac{3}{8}\sin x\cos x+\tfrac{3}{8}x$$

and $$I_6=-\frac{1}{6}\sin^6\cos x+\frac{5}{6}I_4$$

$$=-\frac{1}{6}\cos x\sin^5 x+\frac{5}{6}\left[-\frac{1}{4}\sin^3 x\cos x-\frac{3}{8}\sin x\cos x+\frac{3}{8}x\right]$$

$$=-\frac{1}{6}\cos x\sin^5 x-\frac{5}{24}\sin^3 x\cos x-\frac{5}{16}\sin x\cos x+\frac{5x}{16}$$

$$\therefore\quad\int\sin^6 x\,dx$$

$$=-\frac{1}{6}\cos x\sin^5 x-\frac{5}{24}\sin^3 x\cos x-\frac{5}{16}\sin x\cos x+\frac{5x}{16}+c.$$

(*ii*) **Reduction formula for $\int \cos^n x\,dx$.**

Suppose that $I_n = \int \cos^n x\,dx$.

$$= \int \cos^{n-1} x \cos x\,dx.$$

Write $u = \cos^{n-1} x$ and $\frac{du}{dx} = \cos x$

Then we get,

$\frac{du}{dx} = -(n-1)\cos^{n-2} x \sin x$ and $v = \sin x$

Using now the formula (**1**) for integration by parts, we get

$$I_n = \cos^{n-1} x \sin x - \int \sin x[-(n-1)\cos^{n-2} x \sin x]\,dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x (1-\cos^2 x)\,dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$\therefore \quad I_n = \sin x \cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n.$$

$$\therefore \quad nI_n = \sin x \cos^{n-1} x + (n-1)I_{n-2}$$

$$\therefore \quad I_n = \frac{1}{n}\sin x \cos^{n-1} x + \left(\frac{n-1}{n}\right)I_{n-2}. \qquad \ldots(2)$$

(2) is a *reduction formula* which connects I_n and I_{n-2}. As before,

$I_1 = \int \cos x\,dx = \sin x$ and

$I_0 = \int 1.dx = x$.

Illustrative Example.

Ex. 2. Evaluate $\int \cos^5 x\,dx$.

Sol. $I_5 = \int \cos^5 x\,dx$.

Since I_5 is odd, it is finally connected with I_1. Putting $n = 3$ and 5 in, we get

$$\therefore \quad I_3 = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}I_1$$

$$= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x$$

and

$$I_5 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}I_3$$

$$= \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left[\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x\right]$$

$$= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x$$

$$\therefore \quad \int \cos^5 x\, dx$$

$$= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + c.$$

EXERCISE 5.9.

Integrate the following functions ;

1. $\sin^5 x$; 2. $\cos^6 x$.

ANSWERS

1. $-\frac{1}{5}\sin^4 x\cos x - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x + c.$

2. $\frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{16}\cos x \sin x + \frac{15}{6}x + c.$

5.10. Integral of $\cos^m x$. $\sin^n x$.

In this case, we shall discuss the method of obtaining the integrals of $\cos^m x \sin^n x$ when both m and n are positive integers. If m or n is odd power, then the problem of evaluating.

$\int \cos^m x \sin^n x\, dx$ is extremely easy.

If m is odd, we should put $t = \cos x$ and then evaluate the integral. If n is odd, then put $t = \sin x$ and accordingly evaluate the integral. The following examples will make the procedure clear.

Ex. Evaluate (*i*) $\int \sin^4 x \cos^3 x\, dx$;

(*ii*) $\int \sin^7 x \cos^2 x\, dx$;

Sol. (*i*) We have

$$\int \sin^4 x \cos^3 x\, dx$$

$$= \int \sin^4 x \cos^2 x \cos x\, dx$$

$$= \int \sin^4 x\, (1 - \sin^2 x) \cos x\, dx.$$

Now, put $\sin x = t$

$\therefore \quad \cos x\, dx = dt.$

$$\therefore \quad \int \sin^4 x \cos^3 x\, dx = \int t^4 (1 - t^2)\, dt$$

$$= \int (t^4 - t^6)\, dt$$

$$= \frac{t^5}{5} - \frac{t^7}{7} + c.$$

$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c.$$

(*ii*) We have,

$$\int \sin^7 x \cos^2 x \, dx = \int \cos^2 x \sin^6 x \sin x \, dx$$

$$= \int \cos^2 x \, (1-\cos^2 x)^3 \sin x \, dx.$$

Put $t = \cos x.$

$\therefore \quad dt = -\sin x \, dx.$

$\therefore \quad \sin x \, dx = -dt.$

$\therefore \quad \int \sin^7 x \cos^2 x \, dx$

$$= -\int t^2 (1-t^2)^3 \, dt$$

$$= -\int t^2 (1-t^6+3t^4-3t^2) \, dt$$

$$= \int (t^8-3t^6+3t^4-t^2) \, dt$$

$$= \frac{t^9}{9} - \frac{3t^7}{7} + \frac{3t^5}{5} - \frac{t^3}{3} + c$$

$$= \frac{\cos^9 x}{9} - \frac{3\cos^7 x}{7} + \frac{3}{5}\cos^5 x - \frac{\cos^3 x}{3} + c$$

If both m and n are even, then either convert the integral into integral containing only powers of $\sin x$ or only powers of $\cos x$ and then apply reduction formula to evaluate the integral.

Ex, 1. $\int \cos^4 x \sin^6 x \, dx$

$$= \int (1-\sin^2 x)^2 \sin^6 x \, dx$$

$$= \int (1-2\sin^2 x+\sin^4 x) \sin^6 x \, dx$$

$$= \int \sin^6 x \, dx - 2\int \sin^8 x \, dx + \int \sin^{10} x \, dx.$$

Now, each integral can be evaluated using the reduction formula for the sine function.

5·11. Integration by Partial Fractions.

Consider the expression $\frac{1}{3}\left(\frac{1}{x+2} - \frac{1}{x+5}\right)$

Simplifying it, we get

$$\frac{1}{3}\left[\frac{(x+5)-(x+2)}{(x+2)(x+5)}\right]=\frac{1}{3}\,\frac{3}{(x+2)\,(x+5)}$$

$$=\frac{1}{(x+2)\,(x+5)}$$

Now reversing the steps, we can write,

$$\frac{1}{(x+2)\,(x+5)}=\frac{1}{3}\left(\frac{1}{x+2}-\frac{1}{x+5}\right)$$

i.e.

$$\frac{1}{(x+2)\,(x+5)}=\frac{\left(\frac{1}{3}\right)}{x+2}+\frac{\left(-\frac{1}{3}\right)}{x+5}$$

Here, $\frac{\left(\frac{1}{3}\right)}{x+2}$ and $\frac{\left(-\frac{1}{3}\right)}{x+5}$ are called the *partial fractions of* $\left[\frac{1}{(x+2)(x+5)}\right]$. Thus we have expressed $\left[\frac{1}{(x+2)(x+5)}\right]$ into two partial fractions.

We observe that there are two distinct linear factors in the denominator and the number of partial fractions is also two. Hence we may expect that if there are n distinct linear factors in denominator, then the number of partial fractions will be also n.

The process of finding the integral in certain cases, becomes easy if we make use of partial fractions.

Consider the function

$$F(x)=\frac{2x^2+3x+5}{(x+1)\,(x+2)\,(2x+3)}$$

This function is expressed as a quotient of two polynomials $f(x)$ and $g(x)$ where

$$f(x)=2x^2+3x+5$$

and

$$g(x)=(x+1)\,(x+2)\,(2x+3)$$

The degree of the polynomial $f(x)$ in the numerator is less than the degree of the polynomial $g(x)$ in the denominator. Also, $g(x)$ is a product of three distinct linear factors.

As we expressed $\left[\frac{1}{(x+2)(x+5)}\right]$ into two partial fractions, if we can express similarly $\left[\frac{2x^2+3x+5}{(x+1)(x+2)(2x+3)}\right]$ into three partial fractions ; then the problem of finding its integral will become easy.

Let $$\frac{2x^2+3x+5}{(x+1)\,(x+2)\,(2x+3)}\equiv\frac{A}{(x+1)}+\frac{B}{(x+2)}+\frac{C}{(2x+3)}$$

Then, $\dfrac{2x^2+3x+5}{(x+1)\,(x+2)\,(2x+3)}$

$$=\frac{A\,(x+2)\,(2x+3)+B\,(x+1)\,(2x+3)+C\,(x+1)\,(x+2)}{(x+1)\,(x+2)\,(2x+3)} \quad \ldots(1)$$

$$\therefore \quad 2x^2+3x+5 \equiv A\,(x+2)(2x+3)+B\,(x+1)(2x+3) + C(x+1)(x+2). \quad \ldots(2)$$

This identity must be true for all values of x.

Putting $x+1=0$ in both the sides of (1), we get

$$2(-1)^2+3(-1)+5=A\,(-1+2)(-1+3)+B\times(0)(-2+3) + C\times(0)\times(-1+2),$$

which on simplification gives $A=4$.

Putting $x+2=0$, we get,

$$2(-2)^2+3(-2)+5=A\times(0)\times(-4+3)+B(-2+1)(-4+3) + C\times(-2+1)\times(0)$$

which on simplification gives $B=7$.

Putting $(2x+3)=0$, we get,

$$2\left(-\frac{3}{2}\right)^2+3\left(-\frac{3}{2}\right)+5 = A\times\left(-\frac{3}{2}+2\right)\times 0 + B\times\left(-\frac{3}{2}+1\right)\times 0+C\left(-\frac{3}{2}+1\right)\left(-\frac{3}{2}+2\right)$$

which on simplification gives $C=-20$.

$$\therefore \quad \frac{2x^2+3x+5}{(x+1)(x+2)(2x+3)}=\frac{4}{x+1}+\frac{7}{x+2}+\frac{(-20)}{2x+3} \quad \ldots(3)$$

In this way $\dfrac{2x^2+3x+5}{(x+1)(x+2)(2x+3)}$ is expressed into three partial fractions.

Working rule for finding partial fractions

In order to find the partial fraction corresponding to the factor $x+1$, we had $x+1=0$ *i.e* $x=-1$ in (2) and found the value of A. Instead of this, we follow the following method.

In order to find the partial fraction corresponding to the factor $(x+1)$, let us drop the factor $(x+1)$ from

$$F(x)=\frac{2x^2+3x+5}{(x+1)(x+2\ (2x+3)} \text{ and consider}$$

$$\frac{2x^2+3x+5}{(x+2)(2x+3)}$$

Now put $x+1=0$ (*i.e.*, $x=-1$) in this expression.

Then, we get,

$$\frac{2(-1)^2+3(-1)+5}{(-1+2)(-2+3)}=\frac{4}{1}=4$$

which is the value of A.

Similarly for finding the partial fraction corresponding to $(x+2)$, put $x+2=0$

(*i.e.* $x=-2$) in $\frac{2x^2+3x+5}{(x+1)(2x+3)}$ (here we drop $x+2$)

Then we get,

$$\frac{2(-2)^2+3(-2)+5}{(-2+1)(-4+3)}=\frac{7}{1}=7$$

which is the value of B.

Similarly for finding the partial fraction corresponding to $2x+3$, put $2x+3=0$

$\left(\text{i.e., } x=-\frac{3}{2}\right)$ in $\frac{2x^2+3x+5}{(x+1)(x+2)}$.

Then we get,

$$\frac{2\left(-\frac{3}{2}\right)^2+3\left(-\frac{3}{2}\right)+5}{\left(-\frac{3}{2}+1\right)\left(-\frac{3}{2}+2\right)}=\frac{5}{-1/4}=-20$$

which is the value of C.

Now we can find the integral of

$\frac{2x^2+3x+5}{(x+1)(x+2)(2x+3)}$ very easily.

$$\int\frac{2x^2+3x+5}{(x+1)(x+2)(2x+3)}\,dx=\left\{\left(\frac{4}{x+1}+\frac{7}{x+2}+\frac{-20}{2x+3}\right)\right.dx$$

$$=4\log(x+1)+7\log(x+2)-20\,\frac{\log(2x+3)}{2}+c$$

$$=4\log(x+1)+7\log(x+2)-10\log(2x+3)+c.$$

Illustrative Examples

Ex. 1. Find $\frac{x+3}{(x-1)(x-2)(2x+1)}\,dx$.

Sol. Let $\frac{x+3}{(x-1)(x-2)(2x+1)}\equiv\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{2x+1}$.

Putting $x-1=0$ (*i.e.* $x=1$) in $\left[\frac{x+3}{(x-2)(2x+1)}\right]$,

we get,

$$A=\frac{1+3}{(1-2)(2+1)}=-\frac{4}{3},$$

Putting $x-2=0$ (i.e. $x=2$) in $\left[\frac{x+3}{(x-1)(2x+1)}\right]$

we get,

$$B=\frac{2+3}{(2-1)(4+1)}=\frac{5}{5}=1.$$

Putting $2x+1=0$ (*i.e.* $x=-\frac{1}{2}$) in $\left[\frac{x+3}{(x-1)(x-2)}\right]$

we get,

$$C=\frac{-\frac{1}{2}+3}{(-\frac{1}{2}-1)(-\frac{1}{2}-2)}=\frac{5/2}{15/4}=\frac{2}{3}.$$

$$\therefore \int\frac{x+3}{(x-1)(x-2)(2x+1)}\,dx$$

$$=\int\left[\frac{-4/3}{x-1}+\frac{1}{x-2}+\frac{2/3}{2x+1}\right]dx$$

$$=\frac{-4}{3}\log(x-1)+\log(x-2)+\frac{1}{3}\log(2x+1)+c$$

Ex. 2 Evaluate $\int\frac{x^2-1}{(x^2+1)(x^2+2)(x^2+3)}\,dx$

Sol. Here in place of x^2 we take u and find the partial fraction.

Let $\frac{x^2-1}{(x^2+1)(x^2+2)(x^2+3)}=\frac{u-1}{(u+1)(u+2)(u+3)}$

$$\equiv\frac{A}{u+1}+\frac{B}{u+2}+\frac{C}{u+3}.$$

Putting $u+1=0$ (i.e. $u=-1$) in $\left[\frac{u-1}{(u+2)(u+3)}\right]$

we get,

$$A=\frac{-1-1}{(-1+2)(-1+3)}=\frac{-2}{2}=-1,$$

Putting $u+2=0$, (*i.e.* $u=-2$) in $\left[\frac{u-1}{(u+1)(u+3)}\right]$

we get,

$$B=\frac{-2-1}{(-2+1)(-2+3)}=\frac{-3}{-1}=3.$$

Putting $u+3=0$ [*i.e.* $u=-3$] in $\left[\frac{u-1}{(u+1)(u+2)}\right]$

we get,

$$C=\frac{-3-1}{(-3+1)(-3+2)}=\frac{-4}{2}=-2.$$

$$\therefore \int\frac{x^2-1}{(x^2+1)(x^2+2)(x^2+3)}\,dx=\int\left[\frac{-1}{x^2+1}+\frac{3}{x^2+2}+\frac{-2}{x^2+3}\right]dx$$

$$= -\tan^{-1}x+\frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}-\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+c.$$

Ex. 3. Evaluate $I=\int\frac{4x^3-4x^2-x+3}{2x^2+x-1}\,dx$

Sol. Here the degree of the polynomial in the numerator is larger than the degree of the polynomial in the denominator. Hence by actual division, we get,

$$\frac{4x^3-4x^2-x+3}{2x^2+x-1}=(2x-3)+\frac{4x}{(2x^2+x-1)}.$$

Let $\frac{4x}{2x^2+x-1}\equiv\frac{4x}{(2x-1)(x+1)}=\frac{A}{2x-1}+\frac{B}{x+1}$.

Putting $2x-1=0$ (*i.e.* $x=\frac{1}{2}$) in $\left(\frac{4x}{x+1}\right)$

we get,

$$A=\frac{2}{\frac{1}{2}+1}=\frac{4}{3}.$$

Putting $x+1=0$ *i.e.* $x=-1$ in $\frac{4x}{2x-1}$,

we get,

$$B=\frac{-4}{-2-1}=\frac{4}{3}.$$

$$\therefore\quad\frac{4x}{2x^2+x-1}=\frac{4/3}{2x-1}+\frac{4/3}{x+1}.$$

$$\therefore\quad I=\int\left(2x-3+\frac{4/3}{2x-1}+\frac{4/3}{x+1}\right)dx$$

$$=x^2-3x+\frac{4}{3}\frac{\log|(2x-1)|}{2}+\frac{4}{3}\log|(x+1)|$$

$$=x^2-3x+\frac{2}{3}\log|2x-1|+\frac{4}{3}\log|x+1|+c,$$

Ex. 4. Evaluate $\int\frac{dx}{\sin x+\sin 2x}$.

Sol. Let $I=\frac{dx}{\sin x+\sin 2x}$

$$\therefore\qquad I=\int\frac{dx}{\sin x+2\sin x\cos x}$$

$$=\int\frac{dx}{\sin x(1+2\cos x)}$$

$$=\int\frac{\sin x\,dx}{\sin^2 x(1+2\cos x)}$$

$$= \int \frac{\sin x}{(1-\cos^2 x)(1+2\cos x)}\, dx.$$

Put $\cos x = t$

$\therefore$ $\sin x\, dx = -dt.$

$\therefore$ $I = \int \frac{-dt}{(1-t^2)(1+2t)}.$

Let $\left[\frac{1}{(1-t)(1+t)(1+2t)}\right]$

$$\equiv \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+2t}.$$

Putting $1-t=0$ [*i.e.* $t=1$] in $\left[\frac{1}{(1+t)(1+2t)}\right]$

we get,

$$A = \frac{1}{(1+1)(1+2)} = \frac{1}{6}.$$

Putting $1+t=0$ [*i.e.* $t=-1$] in $\left[\frac{1}{(1-t)(1+2t)}\right]$

we get,

$$B = \frac{1}{(1+1)(1-2)} = -\frac{1}{2}.$$

Putting $1+2t=0$ $\left[i.e.\ t=-\frac{1}{2}\right]$ in $\frac{1}{(1-t)(1+t)}$

we get,

$$C = \frac{1}{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)} = \frac{4}{3}$$

$$I = -\frac{1}{6}\int \frac{dt}{1-t} + \frac{1}{2}\int \frac{dt}{1+t} - \frac{4}{3}\int \frac{dt}{1+2t}$$

$$= \tfrac{1}{6}\log|1-t| + \tfrac{1}{2}\log|1+t| - \tfrac{2}{3}\log(1+2t) + c$$

$$= \tfrac{1}{6}\log|1-\cos x| + \tfrac{1}{2}\log|1+\cos x| - \tfrac{2}{3}\log|1+2\cos x| + c$$

EXERCISE

Integrate the following w.r.t. x

1. (*i*) $\frac{9x+3}{(x-1)(x+2)(x-3)}$;

   (*ii*) $\frac{x^2}{(x+1)(x-2)(x+3)}$;

   (*iii*) $\frac{(x-1)(x-2)}{(x+3)(x+4)(x+5)}$.

2. (*i*) $\frac{5x+2}{x^2-3x+2}$; (*ii*) $\frac{x+12}{x^2-13x+42}$;

   (*iii*) $\frac{2x+5}{6x^2-x-1}$.

3. (*i*) $\frac{5x^2-3}{x^3-x}$; (*ii*) $\frac{x^2+2x-1}{4x^3-x}$;

(*iii*) $\frac{x-1}{x^3+2x^2-8x}$; (*iv*) $\frac{8x+6}{4x^3+8x^2+3x}$;

(*v*) $\frac{2x^2-4x-1}{2x^3-x^2-x}$.

4. (*i*) $\frac{x^2+1}{(2x+1)(x^2-1)}$; (*ii*) $\frac{x^2-3}{x^3-7x+6}$;

(*iii*) $\frac{(x+2)(x+3)}{x^3-6x^2+11x-6}$.

5. (*i*) $\frac{2x^2-1}{(x^2+4)(x^2-5)}$; (*ii*) $\frac{x^2+37}{(x^2-7)(x^2+4)}$;

(*iii*) $\frac{x^2}{(x^2+1)(x^2-2)(x^2+3)}$;

(*iv*) $\frac{x^2}{x^4+5x^2+6}$; (*v*) $\frac{1}{x^4-16}$.

6. (*i*) $\frac{4x^3+2x^2+1}{4x^3-x}$; (*ii*) $\frac{x^3-4x^2+3x+11}{x^2-5x+6}$;

(*iii*) $\frac{x-x^2}{x^2-2x-3}$.

7. (*i*) $\frac{2x}{(x^3+3)(x^2-1)}$; (*ii*) $\frac{x^3}{(x^2+9)(x^2+4)}$;

(*iii*) $\frac{x^3-2x}{(x^2-1)(x^2+5)}$.

8. (*i*) $\frac{\sec^2 x}{(1-\tan^2 x)(2+\tan x)}$;

(*ii*) $\frac{\sin 2x}{\cos^2 x+4\cos x+3}$;

(*iii*) $\frac{\sec^2 x-2\tan^2 x \sec^2 x}{2\tan x-\tan^3 x}$.

9. (*i*) $\frac{1}{\sin x(2+\cos x)}$; (*ii*) $\frac{1}{\sin x-\sin 2x}$;

(*iii*) $\frac{1}{3\sin x+\sin 2x}$; (*iv*) $\frac{(4\sin x+3)\cos x}{5-\sin x-6\cos^2 x}$.

10. (*i*) $\frac{4}{1+3e^x+2e^{2x}}$; (*ii*) $\frac{5e^{3x}-4e^{2x}}{e^{4x}-5e^{2x}+4}$;

(*iii*) $\frac{1}{x\log x[1-(\log x)^2]}$;

(*iv*) $\frac{1}{\log x^x[(\log x)^2-5\log x+6]}$;

(*v*) $\frac{1}{x(1+4x^3+3x^6)}$.

ANSWERS

1. (*i*) $-2\log|x-1|+\log|x+2|+3\log|x-3|+c$.

(*ii*) $\frac{4}{15}\log|x-2|+\frac{9}{10}\log|x+3|-\frac{1}{6}\log|x+1|+c$;

(*iii*) $10\log|x+3|-30\log|x+4|+21\log|x+5|+c$.

2. (*i*) $12\log|x-2|-7\log|x-1|+c$;

(*ii*) $19\log|x-7|-18\log|x-6|+c$;

(*iii*) $\frac{13}{10}\log|2x-1|-\frac{4}{5}\log|3x+1|+c$.

3. (*i*) $\log|x^3(x^2-1)|+c$;

(*ii*) $\log|x|+\frac{1}{8}\log|2x-1|-\frac{7}{8}\log|2x+1|+c$;

(*iii*) $\frac{1}{8}\log|x|-\frac{5}{24}\log|x+4|+\frac{1}{12}\log|x-2|+c$;

(*iv*) $2\log|x|+\log|2x+3|-\log|2x+1|+c$;

(*v*) $\log\left|\frac{x(2x+1)}{x-1}\right|+c$.

4. (*i*) $\frac{1}{3}\log|x-1|+\log|x+1|-\frac{5}{6}\log|2x+1|+c$;

(*ii*) $-\frac{1}{2}\log|x-1|+\frac{1}{5}\log|x-2|+\frac{3}{10}\log|x+3|+c$;

(*iii*) $6\log|x-1|-20\log|x-2|+15\log|x-3|+c$.

5. (*i*) $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{1}{2\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right|+c$;

(*ii*) $\frac{2}{\sqrt{7}}\log\left|\frac{x-\sqrt{7}}{x+\sqrt{7}}\right|-\frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$;

(*iii*) $\frac{1}{6}\tan^{-1}x+\frac{1}{15\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|+\frac{\sqrt{3}}{10}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$;

(*iv*) $-\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$;

(*v*) $\frac{1}{32}\log\left|\frac{x-2}{x+2}\right|-\frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right)+c$;

6. (*i*) $x+\frac{1}{2}\log\left|\frac{(2x+1)(2x-1)^2}{x^2}\right|+c$;

(*ii*) $\frac{x^2}{2}+x-9\log|x-2|+11\log|x-3|+c$;

(*iii*) $\frac{1}{2}\log\left|\frac{x+1}{(x-3)^2}\right|-x+c$.

7. (*i*) Hint: Put $x^2=t$ Ans. $\frac{1}{4}\log\left|\frac{x^2-1}{x^2+3}\right|+c$;

(*ii*) $-\frac{1}{5}\tan^{-1}\left(\frac{x}{2}\right)+\frac{3}{10}\tan^{-1}\left(\frac{x}{3}\right)+c$;

(*iii*) $-\frac{1}{12}\log|x^2-1|+\frac{7}{12}\log|x^2+5|+c$.

8. (*i*) $\frac{1}{2}\log|1+\tan x|-\frac{1}{6}\log|1-\tan x|$

$-\frac{1}{3}\log|2+\tan x|+c$;

(*ii*) $\log \left| \frac{1+\cos x}{(3+\cos x)^3} \right| + c$;

(*iii*) $\frac{1}{2} \log |\tan x| + \frac{3}{4} \log |(\tan^2 x - 2| + c$.

9. (*i*) $\frac{1}{6} \log |1-\cos x| - \frac{1}{2} \log |1+\cos x| + \frac{1}{3} \log |2+\cos x|$

(*ii*) $-\frac{1}{2} \log |1-\cos x| - \frac{1}{6} \log |1+\cos x|$

$+\frac{2}{3} \log |1-2\cos x+] c$;

(*iii*) $\frac{1}{10} \log |1-\cos x| - \frac{1}{2} \log |1+\cos x| + \frac{2}{5} \log |3+2\cos$ $+c$;

(*iv*) $-\log |1-2\sin x| - \frac{1}{3} \log |1+3\sin x| + c$.

10. (*i*) $x+\log(1+e^x)-2\log(1+2e^x)+c$;

(*ii*) $-\frac{7}{3} \log (e^x+2) + \log (e^x-2) - \frac{1}{6} \log(e-1)|$

$+\frac{3}{3} \log |e^x+1| + c$;

(*iii*) $\log \left| \left[\frac{\log x}{\sqrt{1-(\log x)^2}} \right] \right| + c$;

(*iv*) $\frac{1}{6} \log |\log x| - \frac{1}{2} \log |\log x - 2| + \frac{1}{3} \log |\log x - 3| + c$;

(*v*) **Hint :** Put $x^3 = t$

Ans. $\log x + \frac{1}{6} \log |1+x^5| - \frac{1}{2} \log |1+3x^3| + c$.

6

Definite Integrals

6.1. Preliminaries.

The basic idea of *definite integral* was introduced in the eleventh standard and some simple definite integrals were evaluated. Now that we are in position to tackle quite a few difficult problems of indefinite integrals it is but natural for us extend the scope of the topic of definite integrals, Thereby, we hope to solve more complicated problems of definite integrals.

Def. If $\int f(x)\,dx = g(x) + c$, then

$$\int_a^b f(x)\,dx = \Big[\, g(x) + c \,\Big]_a^b$$
$$= [g(b) + c] - [g(a) + c]$$
$$= g(b) - g(a).$$

Here a is called the lower limit and b in called the upper limit of the integral $\int_a^b f(x)\,dx$. We read $\int_a^b f(x)\,dx$ as the integral of $f(x)$, with respect to x, between the limits a and b.

In the above definition, we observe that the constant of integration is conspicuously absent. This is to say that the definite integrals are independent of the constants of integration.

We may rewrite above definition as

If $g'(x) = f(a)$, then

$$\int_a^b f(x)\,dx = \Big[\, g(b) - g(a) \,\Big].$$

Illustrative Examples

Ex .1. Evaluate (i) $\int_0^{\pi/2} \sin x\,dx$, (ii) $\int_a^a \frac{1}{\sqrt{a^2 - x^2}}\,dx$

(*ii*) $\int_2^3 \frac{1}{x^2+5x+6}\,dx.$

Sol. (*i*) $\int_0^{\pi/2} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi/2}$

$= (-\cos \pi/2)-(-\cos 0)$

$= 1$

(*ii*) $\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx = \Big[\sin^{-1}\Big(\frac{x}{a}\Big)\Big]_0^a$

$= \sin^{-1}\Big(\frac{a}{a}\Big) - \sin^{-1}\Big(\frac{0}{a}\Big)$

$= \sin^{-1}(1) - \sin^{-1}(0)$

$= \pi/2.$

(*iii*) By method of partial fraction, we get

$$\frac{1}{x^2+5x+6} = \Big[\frac{1}{(x+2)} - \frac{1}{(x+3)}\Big]$$

$$\therefore \int \frac{1}{x^2+5x+6}\,dx = \int \Big[\frac{1}{x+2} - \frac{1}{x+3}\Big]dx$$

$$= \log\,|\,x+2\,| - \log\,|\,x+3\,| + c$$

$$= \log\Big|\frac{x+2}{x+3}\Big| + c$$

Applying the definition of definite integral, we get

$$\int_2^3 \frac{1}{x^2+5x+6}\,dx = \Big[\log\Big|\frac{x+2}{x+3}\Big|\Big]_2^3$$

$$= \log\Big(\frac{5}{6}\Big) - \log\Big(\frac{4}{5}\Big)$$

$$= \log\Big(\frac{25}{24}\Big).$$

6·2. Rules for evaluating Definite Integrals.

For proving rules of definite integrals, we may state the definition of definite integrals.

Det : If $g'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = \Big[g(x)\Big]_a^b.$$

Rule 1. $\int_a^b \Big[f_1(x)+f_2(x)\Big]dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx.$

Proof. Let g_1 and g_2 be two functions such that

$g'_1(x) = f_1(x)$ and $g'_2(x) = f_2(x)$

clearly, $\int_a^b f_1(x)\,dx = \Big[g_1(x)\Big]_a^b$

and $\int_a^b f_2(x)\,dx = \Big[g_2(x)\Big]_a^b$

Now, we have

$$\frac{d}{dx}\Big[\, g_1(x) + g_2(x) \,\Big] = \frac{d}{dx}\Big[\, g_1(x) \Big] + \frac{d}{dx}\Big[\, g_2(x) \Big]$$

$$\therefore \quad \frac{d}{dx}\Big[\, g_1(x) + g_2(x) \Big] = \Big[\, f_1(x) + f_2(x) \,\Big].$$

$\therefore$ Applying the definition given above, we get

$$\begin{aligned}
&\int_a^b \Big[\, f_1(x) + f_2(x) \,\Big]\, dx \\
&= \Big[\, g_1(x) + g_2(x) \,\Big]_a^b \\
&= [g_1(b) + g_2(b)] - [g_1(a) + g_2(a)] \\
&= [g_1(b) - g_1(a)] + [g_2(b) - g_2(a)] \\
&= \Big[\, g_1(x) \Big]_a^b + \Big[\, g_2(x) \Big]_a^b \\
&= \int_a^b f_1(x)\, dx + \int_a^b f_2(x)\, dx
\end{aligned}$$

$$\therefore \int_a^b \Big[f_1(x) + f_2(x) \Big]\, dx = \int_a^b f_1(x)\, dx + \int_a^b f_2(x)\, dx.$$

Rule 2. $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)\, dx.$

Proof : Let g be a function such that

$$g^1(x) = f(x).$$

clearly $\displaystyle\int_a^b f(x)\, dx = \Big[\, g(x) \,\Big]_a^b$

Now, we have

$$\frac{d}{dx}\Big[kg(x) \Big] = k\,\frac{d}{dx}\Big[\, g(x) \,\Big] = kg^1(x).$$

i.e. $\displaystyle\frac{d}{dx}\Big[kg(x) \Big] = k\, f(x).$

$$\begin{aligned}
\therefore \quad \int_a^b kf(x)\, dx &= \Big[\, k\, g(x) \Big]_a^b \\
&= kg(b) - kg(a) \\
&= k[g(b) - g(a)] \\
&= k\Big[\, g(x) \,\Big]_a^b \\
&= k\int_a^b f(x)\, dx.
\end{aligned}$$

$$\therefore \quad \int_a^b kf(x)dx = k\int_a^b f(x)dx.$$

Rule 3. If $a < c < b$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Proof : Let g be a function such that

$$g^1(x) = f(x)$$

Clearly $$\int_a^b f(x)dx = \Big[\, g(b) - g(a) \,\Big]$$

$$\int_c^c f(x)dx = \Big[\, g(c) - g(a) \,\Big]$$

and $$\int_c^b f(x)dx = \Big[\, g(b) - g(c) \,\Big]$$

Now, we have

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= [\, g(c) - g(a) \,] + [\, g(b) - g(c) \,]$$

$$= \Big[\, g(b) - g(a) \,\Big] = \int_a^b f(x)\,dx$$

$$\therefore \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Rule 4. $\displaystyle\int_a^a f(x)\,dx = 0.$

Proof : Let g be a function such that

$$g^1(x) = f(x).$$

Clearly then

$$\int_a^a f(x)\,dx = \Big[\, g(x) \,\Big]_a^a = g(a) - g(a) = 0.$$

$$\therefore \quad \int_a^a f(x)\,dx = 0.$$

Rule 5. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

Proof. Let g be a function such that

$$g^1(x) = f(x).$$

Then, we have

$$\int_a^b f(x)\,dx = \Big[\, g(x) \,\Big]_a^b$$

$$= g(b) - g(a)$$

and

$$\int_b^a f(x)\,dx = \Big[\, g(x) \,\Big]_b^a$$

$$= g(a) - g(b)$$

$$= -[\, g(b) - g(a) \,]$$

$$= -\int_a^b f(x)\,dx.$$

$$\therefore \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Illustrative Examples

Ex. 1. Evaluate (*i*) $\int_{-1}^{1}(x^2-2x+3)\,dx$;

(*ii*) $\int_0^{\pi/4}\frac{1}{1-\sin x}\,dx$; (*iii*) $\int_\alpha^\beta \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,dx$

Sol. (*i*) $\int_{-1}^{1}(x^2-2x+3)\,dx$

$$=\int_{-1}^{1}x^2\,dx - 2\int_{-1}^{1}x\,dx + 3\int_{-1}^{1}1\,dx$$

$$=\left[\frac{x^2}{3}\right]_{-1}^{1} - 2\left[\frac{x^2}{2}\right]_{-1}^{1} + 3\left[x\right]_{-1}^{1}$$

$$=\left[\frac{1}{3}-\left(\frac{-1}{3}\right)\right] - 2\left[\frac{1}{2}-\frac{1}{2}\right] + 3\left[1-(-1)\right]$$

$$=\frac{2}{3} + 2\times 0 + 3\times 2$$

$$=\frac{2}{3}+6=\frac{20}{3}.$$

(*ii*) $\int_0^{\pi/4}\frac{1}{1-\sin x}\,dx = \int_0^{\pi/4}\frac{1+\sin x}{1-\sin^2 x}\,dx$

$$=\int_0^{\pi/4}\frac{1+\sin x}{\cos^2 x}\,dx = \int_0^{\pi/4}(\sec^2 x + \sec x\tan x)\,dx$$

$$=\int_0^{\pi/4}\sec^2 x\,dx + \int_0^{\pi/4}\sec x\tan x\,dx$$

$$=\left[\tan x\right]_0^{\pi/4} + \left[\sec x\right]_0^{\pi/4}$$

$$=(1-0)+(\sqrt{2}-1)$$

$$=\sqrt{2}$$

(*iii*) $\int_\alpha^\beta \frac{1}{\sqrt{-\alpha\beta-(\alpha+\beta)x-x^2}}\,dx$

$$=\int_\alpha^\beta \frac{1}{\sqrt{\frac{(\alpha+\beta)^2}{4}-\alpha\beta+\left[x^2-(\alpha+\beta)x+\frac{(\alpha+\beta)^2}{4}\right]}}\,dx$$

$$=\int_\alpha^\beta \frac{1}{\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2-\left(x-\frac{\alpha+\beta}{2}\right)^2}}\,dx$$

$$=\left[\sin^{-1}\left\{\frac{x-\frac{\alpha+\beta}{2}}{\left(\frac{\beta-\alpha}{2}\right)}\right\}\right]_\alpha^\beta$$

$$= \left[\sin^{-1}\left(\frac{2x-(\alpha+\beta)}{\beta-\alpha}\right)\right]_{\alpha}^{\beta}$$

$$= \sin^{-1}\left(\frac{\beta-\alpha}{\beta-\alpha}\right) - \sin^{-1}\left(\frac{\alpha-\beta}{\beta-\alpha}\right)$$

$$= \sin^{-1} 1 - \sin^{-1}(-1)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

EXERCISE 6·2

1. Evaluate the following definite integrals.

(*i*) $\int_4^5 (x^2 - x + 4)\,dx$; (*ii*) $\int_0^4 (x^3 - 3x + 1)\,dx$,

(*iii*) $\int_0^1 (3x-2)^5\,dx$; (*vi*) $\int_1^9 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx$

(*v*) $\int_0^2 \frac{dx}{\sqrt[4]{(2+7x)^3}}$; (*vi*) $\int_3^4 \left(\sqrt{x+1} + \sqrt{x-2}\right)dx$

2. (*a*) If $\int_1^9 (3x^2+1)\,dx = 28$, find a.

(*b*) If $\int_1^2 (2x^2 + 5x + c)\,dx = 16$, find c.

(*c*) If $\int_a^3 (x+1)^2\,dx = \frac{56}{3}$, find a.

3. Evaluate the following integrals.

(*i*) $\int_0^{\pi/2} \sin^2 x\,dx$; (*ii*) $\int_0^{\pi/2} \cos^2 x\,dx$.

4. (*i*) $\int_0^5 \frac{dx}{x^2+2x+10}$; (*ii*) $\int_0^1 \frac{dx}{\sqrt{5-2x-x^2}}$

(*iii*) $\int_{a/2}^{a} \frac{dx}{\sqrt{2ax-x^2}}$ (*iv*) $\int_5^6 \sqrt{(x-3)(7-x)}\,dx$

(*v*) $\int_2^5 \frac{x^2+2x-1}{x^3-x}\,dx$ (*vi*) $\int_0^{1/\sqrt{3}} \frac{dx}{(x^2+1)(9x^2+1)}$

(*vii*) $\int_5^{10} \frac{6x}{(x-1)(x-2)}\,dx$.

ANSWERS

1. (*i*) $\frac{119}{6}$; (*ii*) 44, (*iii*) $\frac{-63}{18}$; (*iv*) $\frac{40}{3}$, (*v*) $\frac{4}{7}\left(2 - 2^{1/4}\right)$;

(*vi*) $\frac{2}{3}\left(5^{3/2} + 2^{3/2} - 9\right)$

2. (*a*) 3, (*b*) $\frac{23}{6}$, (*c*) 1. **3.** (*i*) $\pi/4$, (*ii*) $\pi/4$.

4. (*i*) $\frac{\pi}{12}$, (*ii*) $\sin^1\sqrt{\frac{2}{3}} - \sin^1\sqrt{\frac{1}{6}}$, (*iii*) $\pi/6$.

(*iv*) $\pi/3 + \frac{\sqrt{3}}{2}$; (*v*) $\log 5$ (*vi*) $\frac{5\pi}{48}$, (*vii*) $24 \log \frac{4}{3}$.

6.3. Change of Variable

We already know the rule for change of independent variable used in evaluating an indefinite integral, namely

$\int f(x)dx = \int f[\phi(t)]\ \phi^1(t)\ dt + c.$

where $x = \phi(t)$

A similar rule holds for definite integrals as well. We shall only state the rule and omit its proof as it is beyond the scope of this text.

Rule for change of variable.

Let $x = \phi(t)$ be a differentiable function such that $a = \phi(\alpha)$ and $b = \phi(\beta)$, then

$$\int_a^b f(x)\ d\ x = \int_\alpha^\beta f[\phi(t)]\ \phi^1(t)\ dt$$

Illustrative Examples

Ex. 1. Evaluate $\int_0^{\pi/2} \sin^4 x \cos x\ dx$

Sol. Put $\sin x = t$. Then, $\cos x\ dx = dt$,

Also, when $x = 0$, $\sin x = \sin 0 = 0 = t$

and when $x = \pi/2$, $\sin(\pi/2) = 1 = t$.

i.e. when $x = 0$, $t = 0$ and when $x = \pi/2$, $t = 1$.

$$\therefore \int_0^{\pi/2} \sin^4 x \cos x\ dx = \int_0^1 t^4\ dt = \left[\frac{t^5}{5}\right]_0^1$$

$$= \frac{1}{5} - 0 = \frac{1}{5}.$$

Ex. 2. Evaluate $\int_0^1 \frac{1}{x+\sqrt{1-x^2}}\ dx.$

Sol. Put $x = \sin\theta$

$\therefore dx = \cos\theta\ d\ \theta.$

Now, when $x = 0$, $\sin\theta = 0$ and therefore $\theta = 0$.

Also, when $x = 1$, $\sin\theta = 1$ and therefore $\theta = \pi/2$.

$$\therefore \int_0^1 \frac{1}{x+\sqrt{1-x^2}}\ dx = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}\ d\theta.$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{2\cos\theta}{\sin\theta+\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{(\sin\theta+\cos\theta)+(\cos\theta-\sin\theta)}{(\sin\theta+\cos\theta)}\ d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 \cdot d\ \theta + \frac{1}{2}\int_0^{\pi/2} \frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}\ d\ \theta$$

$$= \frac{1}{2}\Big[\ \theta\ \Big]_0^{\pi/2} + \frac{1}{2}\Big[\ \log\Big|\ \sin\theta+\cos\theta\ \Big|\ \Big]_0^{\pi/2}$$

$$\left(\because \frac{d}{d\theta}(\sin\theta+\cos\theta)=\cos\theta-\sin\theta\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{2}(0)$$

$$= \frac{\pi}{4}.$$

Ex. 3. Evaluate $\int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\,d\theta$

Sol. Put $x = \sin^2\theta$

$\therefore dx = 2\sin\theta\cos\theta\,d\theta = \sin 2\theta\,d\theta$. Also, when $\theta=0$, $x=0$ and when $\theta=\pi/4$, $x=\frac{1}{2}$.

$$\therefore \int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\,d\theta = \int_0^{\pi/4} \frac{\sin 2\theta}{(\sin^2\theta)^2+(1-\sin^2\theta)^2}\,d\theta$$

$$= \int_0^{\pi/4} \frac{\sin 2\theta}{2(\sin^2\theta)^2-2\sin^2\theta+1}\,d\theta$$

$$= \int_0^{1/2} \frac{1}{2x^2-2x+1}\,dx$$

$$= \frac{1}{2}\int_0^{1/2} \frac{1}{x^2-x+\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\int_0^{1/2} \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\left(\frac{1}{2}\right)}\tan^{-1}\left\{\frac{\left(x-\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}\right\}\right]_0^{1/2}$$

$$= \frac{1}{2}\cdot\frac{2}{1}\Big[\ \tan^{-1}(2x-1)\ \Big]_0^{1/2}$$

$$= \tan^{-1}0 - \tan^{-1}(-1)$$

$$= 0-(-\pi/4) = \pi/4.$$

Ex. 4. Evaluate $\int_0^{\pi} x(x+\pi)\cos x\,dx$.

Sol. Let $I = \int_0^{\pi} x(x+\pi)\cos x\,dx$.

$$= \int_0^{\pi} x^2\cos x\,dx + \int_0^{\pi} x\,\pi\cos x\,dx.$$

Integrating by parts, we get

$$I = \left\{ \left[x^2 \sin x \right]_0^{\pi} - \int_0^{\pi} 2x \sin x \, dx \right\}$$

$$+ \pi \left\{ \left[x \sin x \right]_0^{\pi} - \int_0^{\pi} \sin x \, dx \right\}$$

$$= 0 - 2\int_0^{\pi} x \sin x \, dx + \pi \times 0 - \pi \int_0^{\pi} \sin x \, dx$$

$$= -2 \left\{ \left[x(-\cos x) \right]_0^{\pi} - \int_0^{\pi} (-\cos x) \, dx \right\}$$

$$- \pi \left[-\cos x \right]_0^{\pi}$$

$$= -2(\pi) - 2\int_0^{\pi} \cos x \, dx - \pi \left[1 + 1 \right]$$

$$= -2\pi - 2\left[\sin x \right]_0^{\pi} - 2\pi$$

$$= -4\pi - \theta$$

$$= -4\pi.$$

Ex. 5. Find the value of $\int_0^1 \frac{x^2 . \tan^{-1} x}{1 + x^2} \, dx$

Sol. Put $x = \tan\theta$, $dx = \sec^2\theta \, d\theta$.

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \pi/4$.

$$\therefore \int_0^1 \frac{x^2 \tan^{-1} x}{1 + x^2} \, dx = \int_0^{\pi/4} \frac{\tan^2\theta . \theta}{\sec^2\theta} \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \theta \tan^2\theta \, d\theta = \int_0^{\pi/4} \theta (\sec^2\theta - 1) \, d\theta$$

$$= \int_0^{\pi/4} \theta \sec^2\theta \, d\theta - \int_0^{\pi/4} \theta \, d\theta$$

$$= \left[\theta . \tan\theta \right]_0^{\pi/4} - \int_0^{\pi/4} \tan\theta \, d\theta - \int_0^{\pi/4} \theta \, d\theta$$

$$= \left[\theta \cdot \tan\theta \right]_0^{\pi/4} - \left[-\log |\sec\theta| \right]_0^{\pi/4} - \left[\frac{\theta^2}{2} \right]_0^{\pi/4}$$

$$= \pi/4 - \log\sqrt{2} - \frac{\pi^2}{32} = \frac{\pi}{4} - \frac{1}{2}\log 2 - \frac{\pi^2}{32}$$

EXERCISE 6·3.

Evaluate the following definite integrals.

1. (*i*) $\int_{-1}^{1} \sqrt{\frac{1-x}{1-x}} \, dx$; (*ii*) $\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

(*iii*) $\int_0^9 \frac{x\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}} \, dx$; (*iv*) $\int_0^1 \frac{dx}{(2-x^2)\sqrt{4-x^2}}$

(*v*) $\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} \, dx.$

2. (*i*) $\int_0^{\pi/2} \frac{\sin dx}{2+\sin^2 x+2\cos^2 x}\, dx$; (*ii*) $\int_{\pi/3}^{\pi/2} \frac{dx}{\sin x+\sin 2x}$;

(*iii*) $\int_0^{\pi/2} \frac{dx}{1+2\sin x+\cos x}$; (*iv*) $\int_0^{\pi/2} \frac{3\sin x\cos x}{5\sin^2 x+2\cos^2 x}\, dx$;

3. (*i*) $\int_{\pi/6}^{\pi/2} x\cos 3x\, dx$; (*ii*) $\int_{-1}^{1} x\tan^{-1} x\, dx$

(*iii*) $\int_0^1 x^2\tan^{-1} x\, dx$; (*iv*) $\int_{\pi/6}^{\pi/3} e^x\sqrt{3}\sin x\, dx$.

4. $\int_0^1 \frac{x(\sin^{-1}x)^2}{\sqrt{1-x^2}}\, dx$.

ANSWERS

1. (*i*) Hint : Put $x=\cos\theta$; Ans. : π ;

(*ii*) Hint : Put $x=\sin\theta$; $I=\int_0^{\pi/2} \frac{d\phi}{1+\sin^2\phi}$

$=\int_0^{\pi/2} \frac{\operatorname{cosec}^2\theta}{2+\cot^2}\cdot\varphi\, d\theta$; Ans : $\frac{\pi}{2\sqrt{2}}$

(*iii*) Hint : Put $x^2=a^2\cos 2\theta$, Ans : $\frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$

(*iv*) Hint : Put $x=2\sin\theta$; Ans. $\frac{1}{4}\log\left(2+\sqrt{3}\right)$;

(*v*) $\pi/4$.

2. (*i*) $\log\frac{4}{3}$; (*ii*) $\frac{4}{3}\log 2-\frac{1}{2}\log 3$; (*iii*) $\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}$;

(*iv*) $\frac{1}{2}\log\frac{5}{2}$.

3. (*i*) $\frac{2}{9}\left(1+\frac{\pi}{2}\right)$ (*ii*) $\frac{\pi}{2}-1$; (*iii*) $\frac{\pi}{8}-\frac{1}{6}$

(*iv*) $\frac{1}{4}e^{\pi/\sqrt{3}}$.

4. $\pi-2$.

6·4. Properties of Definite Integrals

There are certain nice properties of definite integrals — the knowledge of which can certainly help us to reduce the computation work while evaluating particular definite integrals. We state below some of the interesting properties.

I. $\int_a^b f(x)\, dx = \int_a^b f(y)\, dy$

Proof. Let g be a function such that $g^1(x) = f(x)$.

Then, $\int_a^b f(x)\, dx = \Big[g(x)\Big]_a^b = g(b) - g(a)$.

Also, $g^1(y) = f(y)$,

$\therefore \int_a^b f(y)\, dy = \Big[g(y)\Big]_a^b = g(b) - g(a)$

$\therefore \int_a^b f(x)\, dx = \int_a^b f(y)dy$.

The above property shares that the value of the definite integral does not depend on the representation (x or y) that we give to the variable, but it only depends upon the limits of integration.

II. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$

Proof. Put $a + b - x = t.$

$\therefore\ dx = dt$ or $dx = -dt$

Now, when $x = a$, $t = b$ and when $x = b$, $t = a$,

$\therefore\ \int_a^b f(a + b - x)\,dx = -\int_b^a f(t)\,dt$

$= \int_a^b f(t)\,dt$ (by Rule 5)

$= \int_a^b f(t)\,dt$ by property I

$= \int_a^b f(x)\,dx$

$\therefore\ \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Cor. $= \int_o^a f(x)\,dx = \int_o^a f(a-x)\,dx,$

III. $\int_o^{2a} f(a)\,dx = \int_o^a f(x)\,dx + \int_o^a f(a-x)\,dx.$

Proof. We have, by rule 3,

$$\int_0^{2a} f(x)\,dx = \int_o^a f(x)\,dx + \int_a^{2a} f(x)\,dx. \qquad (1)$$

Let $I = \int_a^{2a} f(x)\,dx$

Put $a - x = -t.$

$\therefore\ dx = dt$

when $x = a$, $t = 0$ and when $x = 2a$, $t = a$

$\therefore\ I = \int_a^{2a} f(x)\,dx = \int_o^a f(a-t)\,dt$

$= \int_o^a f(a-x)\,dx$ by property I

$\therefore$ (1) becomes,

$\int_0^{2a} f(x)\,dx = \int_o^a f(x)\,dx + \int_o^a f(a-x)\,dx.$

IV. $\int_o^a f(x)\ dx = 2\int_o^a f(x)\ dx$, if f is even

$= 0$, if f is odd.

Proof. (*i*) Let f be a even function.

Then, $f(-x) \equiv f(x)$

Now, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{o} f(x)\ dx + \int_o^a f(x)\ dx. \qquad (2)$$

Let $I = \int_{-a}^{o} f(x)\ dx.$

Put $x = -t.$

$\therefore\ dx = -dt.$

Clearly, when $x = -a$, $t = a$ and when $x = 0$, $t = 0$,

$$\therefore I = \int_{-a}^{o} f(x)dx = -\int_a^o f(-t)\ dt$$

$$= \int_o^a f(-t)\ dt \qquad \text{(Rule 5)}$$

$$= \int_o^a f(t)\ dt \qquad (\because f \text{ is even})$$

$$= \int_o^a f(x)\ dx) \qquad \text{(Property I)}$$

Substituting this value of I in (2), we get

$$\int_{-a}^{a} f(x)\ dx = 2\int_o^a f(x)\ dx.$$

(*ii*) Now, let f be a odd function

Then $f(-x) = -f(x)$.

We have $\int_{-a}^{a} f(x)\ dx = \int_{-a}^{o} f(x)\ dx + \int_o^a f(a)\ dx \qquad (3)$

Let $I = \int_{-a}^{o} f(x)\ dx.$

Put $x = -t$

$\therefore\ dx = -dt$

Clearly, when $x = -a$, $t = a$ and when $x = 0$, $t = 0$.

$$\therefore I = \int_{-a}^{o} f(x)\ dx$$

$$= \int_a^o f(-t)\ dt$$

$$= -\int_o^a f(-t)\ dt \qquad \text{(Rule 5)}$$

$$= -\int_o^a f(t)\ dt) \qquad (\because f \text{ is odd})$$

$$= -\int_o^a f(x)\ dx \qquad \text{(Property I)}$$

$$\therefore \quad I = \int_a^o f(x)\,dx \quad = -\int_o^a f(x)\,dx$$

Putting this in (3), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{o} f(x)\,dx + \int_o^a f(x)\,dx$$

$$= -\int_o^a f(x)\,dx + \int_o^a f(x)\,dx$$

$$= 0.$$

We shall now show how these properties can be used skillfully to simply the computations of some definite integrals with the help of the following examples.

Illustrative Examples

Ex. 1. Evaluate $\int_{,} x^2 \sqrt{3-x}\,dx$

Sol. $\int_o^3 x^2 \sqrt{3-x}\,dx$

$= \int_o^3 (3-x)^2 \sqrt{3-(3-x)}\,dx$ (Property II)

$= \int_o^3 (9 - 6x + x^2) \sqrt{x}\,dx.$

$= \int_0^3 (9x^{1/2} - 6x^{3/2} + x^{5/2})\,dx$

$= \left[\frac{9x^{3/2}}{(3/2)} - \frac{6x^{5/2}}{(5/2)} + \frac{x^{7/2}}{(7/2)}\right]_0^3$

$= 18\sqrt{3} - \frac{108}{5}\sqrt{3} + \frac{54}{7}\sqrt{3}$

$= \frac{144}{35}\sqrt{3}.$

Ex. 2. Evaluate $\int_0^\pi x \sin^2 x\,dx.$

Sol. Let $I = \int_0^\pi x \sin^2 x\,dx.$

$I = \int_0^\pi (\pi - x) \sin^2(\pi - x)\,dx$ (Property II)

$= \int_0^\pi (\pi - x) \sin^2 x\,dx$

$= \pi \int_0^\pi \sin^2 x\,dx - \int_0^\pi x \sin^2 x\,dx$

$\therefore \quad I = \pi \int_0^\pi \sin^2 x\,dx - I$

$$\therefore\ 2I = \pi \int_0^{\pi} \sin^2 x\, dx$$

$$\therefore\ I = \frac{\pi}{2}\int_0^{\pi} \frac{1-\cos 2x}{2}\, dx$$

$$\therefore\ I = \frac{\pi}{4}\int_0^{\pi} (1-\cos 2x)\, dx$$

$$= \frac{\pi}{4}\left[x - \frac{\sin^2 x}{2}\right]_0^{\pi}$$

$$= \frac{\pi}{4}\left[\pi - 0\right] = \frac{\pi^2}{4}.$$

Ex. 3. Evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx.$

Sol. Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx.$

$$\therefore\ I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2-x)} + \sqrt{\cos(\pi/2-x)}}\, dx \text{ (Property II)}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$$

$$\therefore\ 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx.$$

$$= \int_0^{\pi/2} 1\, dx$$

$$= \frac{\pi}{2}.$$

$$\therefore\ 2I = \frac{\pi}{2}$$

$$\therefore\ I = \frac{\pi}{4}$$

Ex. 4. Evaluate $\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\, dx.$

Sol. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx.$

$$\therefore \quad I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$$

$$= \int_0^{\pi} \frac{(\pi - x) \sin x}{1+\cos^2 (\pi - x)} dx$$

$$= \int_0^{\pi} \frac{(\pi - x) \sin x}{1+\cos^2 x}\, dx$$

$$= \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

$$= \pi . \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx - I.$$

$$\therefore 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

$$\therefore \quad I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx.$$

Now, put $\cos x = t$.

$\therefore \sin x\, dx = -dt$

Also, when $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$.

$$\therefore \quad I = \frac{\pi}{2} \int_1^{-1} \frac{-dt}{1+t^2}$$

$$= \frac{\pi}{2} \int_{-1}^{1} \frac{1}{1+t^2} dt$$

$$= \frac{\pi}{2} \Big[\tan^{-1} t \Big]_{-1}^{1}$$

$$= \frac{\pi}{2} \Big[\tan^{-1} (1) - \tan^{-1} (-1) \Big]$$

$$= \frac{\pi}{2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = \frac{\pi^2}{4}.$$

Ex. 5. Evaluate (*i*) $\int_{-1/2}^{1/2} \frac{1+x^3}{4-x^2}\, dx$ (*ii*) $\int_{-\pi/2}^{\pi/2} x^3 \sin^2 x dx$

Sol.

(*i*) Let $I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1+x^3}{4-x^2}\, dx$

Clearly, $I = \int_{-1/2}^{1/2} \frac{1}{4-x^2}\, dx + \int_{-1/2}^{1/2} \frac{x^3}{4-x^2} dx$

$= I_1 + I_2$ (say)

Now $\frac{1}{4-x^2}$ is a even function

$$\therefore I_1 = \int_{-1/2}^{1/2} \frac{1}{4-x^2}\, dx = 2 \int_0^{1/2} \frac{1}{4-x^2} dx = \frac{2}{4} \left[\log \left| \frac{2+x}{2-x} \right| \right]_0^{1/2}$$

$$= \frac{1}{2} \log \left(\frac{5}{3} \right).$$

But, $\frac{x^3}{4-x^2}$ is an odd function.

$\therefore I_2 = \int_{-1/2}^{1/2} \frac{x^3}{4-x^2}\, dx = 0.$

$\therefore \int_{-1/2}^{1/2} \frac{1+x^3}{4-x^2}\, dx = \frac{1}{2} \log \left(\frac{5}{3}\right).$

(*ii*) Since $x^3 \sin^2 x$ is an odd function,

$$\int_{-\pi/2}^{\pi/2} x^3 \sin^2 x\, dx = 0.$$

EXERCISE 6·4.

Evaluate the following definite Integrals.

1. (*i*) $\int_0^1 x^3 (1-x^2)^{3/2}\, dx$; (*ii*) $\int_0^a x^2 (a^2-x^2)^{3/2}\, dx$;

(*iii*) $\int_0^1 x^2 (1-x)^{1/2}\, dx$; (*iv*) $\int_0^1 x^2 (1-x)^{5/2}\, dx$;

(*v*) $\int_0^3 x^2 (3-x)^{1/2}\, dx.$

2. (*i*) $\int_0^\pi \frac{x \sin x}{1+\sin x}\, dx$; (*ii*) $\int_0^\pi \frac{x}{1+\sin x}\, dx$;

(*iii*) $\int_0^{\pi/2} \frac{x \sec x}{1+\tan x}\, dx$; (*iv*) $\int_0^\pi \frac{x \sec x \tan x}{2+\tan^2 x}\, dx$;

(*v*) $\int_0^\pi \frac{x \sin x}{1+\cos^2 x}\, dx$;

3. (*i*) $\int_0^4 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\, dx$; (*ii*) $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\, dx$;

(*iii*) $\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}}\, dx$; (*iv*) $\int_0^{\pi/2} \frac{\cos^2 x}{\sin x+\cos x}\, dx$;

(*v*) $\int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\, dx$; (*vi*) $\int_0^{\pi/2} \frac{\tan x}{1+\tan x}\, dx$

(*vii*) $\int_0^{\pi/2} \frac{1}{1+\tan x}\, dx$; (*viii*) $\int_0^1 \frac{1}{x+\sqrt{1-x^2}}\, dx$;

(*x*) $\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}\, dx.$

4. (*i*) $\int_{-1/2}^{1/2} \frac{1+x^3}{1-x^2}\, dx$; (*ii*) $\int_{-\pi/4}^{\pi/4} \tan^3 x \sec x\, dx$;

(*iii*) $\int_{-\pi/2}^{\pi/2} \frac{x^5-x}{\sin^4 x+x^6}\, dx$; (*iv*) $\int_{-5}^{5} \frac{x^3}{7+9x^2}\, dx$;

(*v*) $\int_{-\pi/2}^{\pi/2} \frac{\sec^2 x+\tan^4 x}{3x^5+5x^2}\, dx.$

5. (*i*) $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\, dx$; (*ii*) $\int_1^3 \frac{\sqrt{x}}{\sqrt{4-x}+\sqrt{x}}\, dx$;

(*iii*) $\int_{\pi/6}^{\pi/3} x \cos 3x\, dx$; (*iv*) $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}}\, dx$;

(*v*) $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cos x}}\, dx$;

ANSWERS

1. (*i*) $\frac{2}{35}$; (*ii*) $\frac{\pi a^6}{32}$; (*iii*) $\frac{16}{105}$; (*iv*) $\frac{16}{693}$; (*v*) $\frac{144\sqrt{3}}{35}$,

2. (*i*) $\pi\left(\frac{\pi}{2}-1\right)$; (*ii*) $\pi\left(\frac{\pi}{2}-1\right)$; (*iii*) $\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$;

(*iv*) $\frac{\pi^2}{1}$; (*v*) $\frac{\pi^2}{4}$.

3. (*i*) **2**; (*ii*) $\frac{\pi}{4}$; (*iii*) $\frac{\pi}{4}$; (*iv*) $\frac{1}{\sqrt{2}}\log(1+\sqrt{2})$;

(*v*) $\frac{\pi}{4}$; (*vi*) $\frac{\pi}{4}$; (*vii*) $\frac{\pi}{4}$; (*viii*) $\frac{\pi}{4}$; (*ix*) $\frac{\pi}{4}$.

4. (*i*) log 3; (*ii*) 0; (*iii*) 0; (*iv*) 0; (*v*) 0.

5. (*i*) $\frac{1}{2}$; (*ii*) 1; (*iii*) $\frac{2}{9}\left(\frac{\pi}{2}+1\right)$; (*iv*) $\frac{\pi}{12}$; (*v*) $\frac{\pi}{12}$.

6·5. Reduction formulae

We have already studied the reduction formulae for the indefinite integrals $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$ where n is a positive integer. Using these formulae, we shall write the corresponding reduction formulae for the definite integrals.

We know that

$$\int \sin^n x\,dx = \frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\therefore \int_0^{\pi/2} \sin^n x\,dx = \left[\frac{\sin^{n-1} x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx.$$

But, $\left[\frac{\sin^{n-1} x\cos x}{n}\right]_0^{\pi/2} = 0.$

$$\therefore \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx.$$

Further, we are aware that

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^n(\pi/2-x)\,dx.$$

$$= \int_0^{\pi/2} \cos^n x\,dx.$$

$\therefore$ We get the reduction formula as

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx.$$

It is clear that every time we use the reduction formula, the power of sin x gets reduced from n to $(n-2)$. Therefore the formula can be successively used till the power of *sin* x reduces to either 1 or zero depending on whether n is odd or even.

Ex. 1. Evaluate (*i*) $\int_0^{\pi/2} \sin^8 x\, dx$; (*ii*) $\int_0^{\pi/2} \sin^8 x\, dx$

Sol. We know the reduction formula

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} dx.$$

(*i*) $\therefore$ Using this successively, we get

$$\int_0^{\pi/2} \sin^7 x\, dx = \frac{6}{7} \int_0^{\pi/2} \sin^5 x\, dx$$

$$= \frac{6}{7}\left[\frac{4}{5} \int_0^{\pi/2} \sin^3 x\, dx\right]$$

$$= \frac{24}{35}\left[\frac{2}{3} \int_0^{\pi/2} \sin x\, dx\right]$$

$$= \frac{16}{35} \int_0^{\pi/2} \sin x\, dx$$

$$= \frac{16}{35}.$$

(*ii*) Applying successively the reduction formula, we get

$$\int_0^{\pi/2} \sin^8 x\, dx = \frac{7}{8} \int_0^{\pi/2} \sin^6 x\, dx$$

$$= \frac{7}{8}\left[\frac{5}{6} \int_0^{\pi/2} \sin^4 x\, dx\right]$$

$$= \frac{35}{48}\left[\frac{3}{4} \int_0^{\pi/2} \sin^2 x\, dx\right]$$

$$= \frac{35}{64}\left[\frac{1}{2} \int_0^{\pi/2} 1\, dx\right]$$

$$= \frac{35}{128} \frac{\pi}{2}$$

$$= \frac{35\pi}{256}$$

Ex. 2. Evaluate (*i*) $\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$

(*ii*) $\int_0^1 x^4 \sqrt{1-x^2}\, dx.$

Sol. (*i*) $\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx.$

$$= \int_0^{\pi/2} \sin^4 x\,(1 - \sin^2 x)\, dx$$

$$= \int_0^{\pi/2} \sin^4 x\, dx - \int_0^{\pi/2} \sin^6 x\, dx,$$

$$\text{Now, } \int_0^{\pi/2} \sin^4 x\, dx = \frac{3}{4} \int_0^{\pi/2} \sin^2 x\, dx$$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot \int_0^{\pi/2} 1.dx$$

$$= \frac{3}{8} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi}{16}.$$

Similarly applying once again reduction formula, we get

$$\int_0^{\pi/2} \sin^6 x\, dx = \frac{5}{6} \cdot \int_0^{\pi/2} \sin^4 x\, dx$$

$$= \frac{5}{6} \cdot \left(\frac{3\pi}{16}\right) \quad \text{(from above)}$$

$$= \frac{5\pi}{32}.$$

$$\therefore \int_0^{\pi/2} \sin^4 x \cos^2 x\, dx = \frac{3\pi}{16} - \frac{5\pi}{32}$$

$$= \frac{6\pi - 5\pi}{32} = \frac{\pi}{32}.$$

(*ii*) Put $x = \sin\theta$.

$\therefore dx = \cos\theta\, d\theta$.

Clearly, when $x = 0$, $\sin\theta = 0$ and therefore $\theta = 0$.

Similarly, when $x = 1$, $\sin\theta = 1$ and therefore $\theta = \pi/2$.

$$\therefore \int_0^1 x^4 \sqrt{1-x^2}\, dx$$

$$= \int_0^{\pi/2} \sin^4\theta \sqrt{1-\sin^2\theta} \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta$$

$$= \frac{\pi}{32}, \text{ calculated in example (1)}$$

$$\therefore \int_0^1 x^4 \sqrt{1-x^2}\, dx = \frac{\pi}{32}.$$

Remark. In case one wants to write the value of, $\int_0^{\pi/2} \sin^n x\, dx$ or $\int_0^{\pi/2} \cos^n x\, dx$ directly, one may remember the following formula.

(*i*) $$\int_0^{\pi/2} \cos^n x\, dx = \int_0^{\pi/2} \sin^n x\, dx$$

$$= \frac{(n-1)(n-3)\ldots 5\cdot 3\cdot 1}{n(n-2)(n-4)\ldots 4\cdot 2} \cdot \frac{\pi}{2} \text{ when } n \text{ is even}$$

(*ii*) $\int_0^{\pi/2} \cos^n x\, dx = \int_0^{\pi/2} \sin^n x\, dx$

$= \dfrac{(n-1)(n-3)\ldots 4\cdot 2\cdot}{n(n-2)(n-4)\ldots 5\cdot 3\cdot 1}$ when n is odd.

EXERCISE 6·5

Prove the following :

1. $\int_0^{\pi/2} \cos^7 x\, dx = \dfrac{16}{35}$; 2. $\int_0^{\pi/2} \cos^4 x\, dx = \dfrac{3\pi}{16}$.
3. $\int_0^{\pi/2} \cos^6 x\, dx = \dfrac{5\pi}{32}$; 4. $\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx = \dfrac{2}{35}$.
5. $\int_0^{\pi/2} \sin^6 x \cos^2 x\, dx = \dfrac{5\pi}{256}$;
6. $\int_0^{\pi/2} \sin^4 x \cos^4 x\, dx = \dfrac{3\pi}{256}$;
7. $\int_0^{\pi/2} \sin^4 x \cos^5 x\, dx = \dfrac{8}{315}$;
8. $\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx = \dfrac{2}{35}$;
9. $\int_0^{\pi/2} \sin^3 x \cos^3 x\, dx = \dfrac{1}{12}$;
10. $\int_0^{\pi/2} \sin^5 x \cos^5 x\, dx = \dfrac{1}{60}$;
11. $\int_0^1 x^6 \sqrt{1-x^2}\, dx = \dfrac{5\pi}{256}$;
12. $\int_0^1 x^5 \sqrt{1-x^2}\, dx = \dfrac{8}{105}$.

6·6. Definite Integral as a Limit of a Sum.

We have *intuitive idea of what is meant* by the area of a plane geometrical figure. It is the positive real number associated with the region bounded by the closed geometrical figure. For example, the area of a rectangle is the product of its length and breadth, the area of a triangle is half the product of the length of its base and the corresponding altitude.

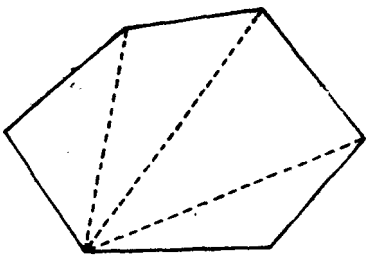

Fig. 6·1.

The area of a polygon can be defined as the sum of the areas of the triangles into which it is decomposed ; and it is independent of how the polygon is decomposed into triangles.

Now, let $y = f(x)$ be any continuous function defined on the closed interval $[a, b]$. For the sake of simplicity, we shall assume that $f(x) \geqslant 0$ for all x in $[a, b]$.

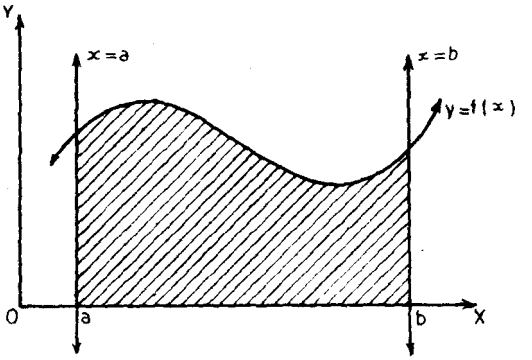

Fig. 6·2.

Consider the area bounded by the curves $y = f(x)$, x-axis, $x = a$ and $x = b$, shown in Fig. 6·2. by the shaded portion.

This area is defined as the definite integral $\int_a^b f(x)\,dx$. We shall compute this by a slightly different method by finding the limit of a sum. We use *limiting process* to find the area.

Consider the *subdivision* of the interval $[a, b]$ given by

$a = x_0 < x_1 < x_2 \ldots < x_{i-1} < x_i < \ldots < x_n = b$

which gives rise to n subintervals, namely

$[x_0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n]$.

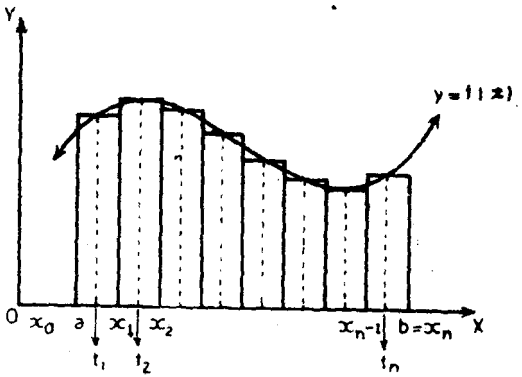

Fig. 6·3.

Select now a point t_i from the ith subinterval $[\, x_{i-1}, x_2 \,]$; $i = 1,2,3, \ldots n$. Clearly we have

$$a = x_0 \leqslant t_1 \leqslant x_1 \leqslant t_2 \leqslant x_2 \ldots < x_{n-1} < t_n < x_n = b$$

Let δ_i be the length of the ith subinterval, $i = 1, 2, \ldots n$.

The norm δ of the subdivision is defined as the largest of the values $\delta_1, \delta_2, \ldots, \delta_n$. Clearly, as $\delta \to 0$, $\delta_i \to 0$.

Now, on each subinterval (on X-axis) as a base, we erect a rectangle whose height is the value of the function f at the point selected from that interval. [*i.e.* $f(t_1)$, $f(t_2)$. ... , $f(t_n)$ are the heights of the respective rectangles] as shown in Fig. 6·3.

Define s_n, the sum of the areas of these rectangles. Then, clearly,

$$s_n = f(t_1)\, \delta_1 + f(t_2)\, \delta_2 \ldots + f(t_n)\, \delta_n.$$

However, these rectangles do not cover the region whose area is under consideration. But, we observe that s_n depends on

(*i*) the number n of subintervals,

(*ii*) the choice of the points $x_1, x_2, \ldots, x_{n-1}$ of the subdivision

and (*iii*) the choice of the points $t_1, t_2, \ldots, t_n$.

It is evident that the sum of the areas of these rectangles, s_n is an approximation of the area under the curve. A better approximation can be obtained with more rectangles each with a smaller base and thereby the upper base of each rectangle would lie quite close to the curve. Thus, as we go on decreasing the base width of the rectangles, we go on making better approximations. This is to *say* that in the limit, the sum of the areas of the rectangles is equal to the area under the curve.

$$\therefore \text{ The area under the curve } y = f(x)$$

$$= \int_a^b f(x)\, dx$$

$$= \lim_{\delta \to 0} s_n$$

$$= \lim_{\delta \to 0} [\, f(t_1)\, \delta_1 + f(t_2)\, \delta_2 + \ldots + f(t_n)\, \delta_n].$$

Note that the above limit is found to be independent of the choice of the points t's.

This method, however, is rather difficult to use in practice. Therefore, by making some *special choice*, we can evaluate the value of the *definite integral* (or the area *under* the curve) comfortably.

We shall *assume* that

(*i*) The subdivision divides the interval $[a, b]$ into n sub intervals of equal length $\frac{b-a}{n} = h$.

Clearly, then the subdivision of $[a, b]$ is given by

$a, a + h, a + 2h, \ldots, a + nh$ where the norm of the subdivision is h.

It is evident that as $h \to 0$, $n \to \infty$ and conversely.

(*ii*) Instead of choosing any arbitrary point t_i from the ith subinterval, we choose $t_i = a + ih$.

With this we get

$$s_n = f(a + h)\,h + f(a + 2h)\,h + \ldots + f(a + nh)\,h$$

$$= \sum_{i=1}^{n} h\,f(a + ih)$$

Clearly, we have

$$\int_a^b f(x)\,dx = \lim_{h \to 0} \left[\sum_{i=1}^{n} h\,f(a + ih) \right]$$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} h\,f(a + ih) \right].$$

Note that $h = \frac{b-a}{n}$ and

as $h \longrightarrow 0$, $n \longrightarrow \infty$ and conversely, as $n \longrightarrow \infty$, $h \longrightarrow 0$.

We shall now show how to evaluate the definite integral as the limit of a sum.

Illustrative Examples

Ex. 1. Evaluate $\int_0^1 x\,dx$ as a limit of the sum.

Sol. Let $f(x) = x$. Clearly f is continuous on the closed interval $[0, 1]$.

Divide the interval $[0,1]$ into n subintervals each of length h at the points given by

$0, 0 + h, 0 + 2h, \ldots\ldots, 0 + nh = 1$

or $0, h, 2h, 3h, \ldots\ldots, nh.$

$$\therefore\ h = \frac{1 - 0}{n} = \frac{1}{n},$$

Clearly, as $h \to 0$, $n \to \infty$

$$\therefore \int_0^1 x\,dx = \int_0^1 f(x)\,dx$$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} h\, f(ih) \right]$$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} h\,(ih) \right] \qquad (\because f(x) = x)$$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} \frac{1}{n^2} i \right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{n^2} \sum_{i=1}^{n} i \right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{n^2} \frac{n(n+1)}{2} \right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{2} \left[1 + \frac{1}{n} \right] \right]$$

$$= \frac{1}{2}$$

$$\int_0^1 x\,dx = \frac{1}{2}.$$

Ex. 2. Evaluate $\int_1^3 x^2\,dx$ as a limit of the sum.

Sol. Divide the subinterval [1,3] into n subintervals, each of length h at the points.

$1,\ 1+h,\ 1+2h,\ 1+3h,\ \ldots\ldots,\ 1+nh=3.$

$$\therefore h = \frac{3-1}{n} = \frac{2}{n}$$

Clearly, as $h \to 0$, $n \to \infty$.

Let now $f(x) = x^2$.

$$f(1+ih) = (1+ih)^2 = 1 + 2ih + i^2h^2$$

$$= 1 + 2i\frac{2}{n} + i^2\frac{4}{n^2}$$

$$= \left(1 + 4\frac{i}{n} + 4\frac{i^2}{n^2}\right)$$

$$\int_1^3 x^2\,dx = \int_1^3 f(x)\,dx$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} h\,f(1+ih)\right]$$

$$= \lim_{n=\infty}\left[\sum_{i=1}^{n}\frac{2}{n}\left(1+\ 4\frac{i}{n}+4\ \frac{i^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{2}{n}\left(\ n\ +\frac{4}{n}\cdot\sum_{i=1}^{n} i\ +\ \frac{4}{n^2}\sum_{i=1}^{n} i^2\ \right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{2}{n}\left\{n\ +\ \frac{4}{n}\frac{n(n+1)}{2}\ +\ \frac{4}{n^2}\cdot\ \frac{n(n+1)\,(2n+1)}{6}\right\}\right]$$

$$= \lim_{n\to\infty}\left[\ 2+4.\left(\ 1\ +\ \frac{1}{n}\right)+\ \frac{4}{3}\cdot\ \left(\ 1\ +\ \frac{1}{n}\right)\left(\ 2\ +\ \frac{1}{n}\right)\right]$$

$$= 2 + 4 + \frac{8}{3} = 6 + \frac{8}{3} = \frac{26}{3}\cdot$$

Ex. 3. Evaluate $\int_0^1 e^{\,x}\,dx$ as a limit of the sum.

Sol. Divide the subinterval [0, 1] into n subintervals, each of length h at the points o, h, $2h$, 3h, ... $nh = 1$.

$$\therefore\ h = \frac{1-0}{n} = \frac{1}{n}\cdot$$

Clearly, as $h \to 0$, $n \to \infty$.

Now, let $f(x) = e^x$,

$\therefore f(ih) = e^{ih}$,

$$\therefore \int_o^1 e^x\,dx = \int_o^1 f(x)\,dx$$

$$= \lim_{h\to o}\left[\sum_{i=1}^{n} h\,f(ih)\right]$$

$$= \lim_{h\to o}\left[\sum_{i=1}^{n} h\,e^{ih}\right]$$

$$= \lim_{h\to o}\left[\ h\left\{e^h\ +\ e^{2h}\ +\ e^{3h}\ +\ \ldots\ +\ e^{\ nh}\right\}\right]$$

$$= \lim_{h\to o}\left[\ h\ e^h\left\{\ 1\ +\ e^{\,h}\ +\ e^{\,2h}\ +\ +\ \ldots e^{(n-1)h}\ \right\}\right]$$

$$= \lim_{h \to 0} \left[h\, e^h \left\{ \frac{e^{nh} - 1}{e^h - 1} \right\} \right]$$

$$= \lim_{h \to 0} \left[e^h (e^{nh} - 1) \cdot \left(\frac{h}{e^h - 1} \right) \right]$$

$$= \lim_{h \to 0} \left[e^h (e^1 - 1) \left(\frac{h}{e^h - 1} \right) \right] \qquad (\because nh = 1)$$

$$= (e-1) \times \lim_{h \to 0} \left[e^h \left(\frac{h}{e^h - 1} \right) \right]$$

$$= (e-1) \times 1 = e - 1.$$

Ex. 4. Evaluate $\int_0^2 (3x + 4)\, dx$ as the limit of a sum.

Sol. Let $f(x) = 3x + 4$ for $0 \leqslant x \leqslant 2$.

Divide the interval $[0, 2]$ into n subintervals each of length h at the points $0, h, 2h, 3h, \ldots, nh = 2$.

$\therefore \quad h = \frac{2}{n}$ so that as $h \to 0$, $n \to \infty$,

Clearly, $f(ih) = (3\, i\, h + 4)$; $i = 1, 2, \ldots, n$.

$$\int_0^2 (3x+4)\, dx = \int_0^2 f(x)\, dx$$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} h\, f(ih) \right]$$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} \frac{2}{n} (3ih + 4) \right]$$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} \frac{2}{n} \left(3\, i. \frac{2}{n} + 4 \right) \right]$$

$$= \lim_{n \to \infty} \left[\frac{2}{n} \left\{ \frac{6}{n} \sum_{i=1}^{n} i + 4n \right\} \right]$$

$$= \lim_{n \to \infty} \left[\frac{2}{n} \left\{ \frac{6}{n} \cdot \frac{n(n+1)}{2} + 4^n \right\} \right]$$

$$= \lim_{n \to \infty} \left[6 \left(1 + \frac{1}{n} \right) + 8 \right]$$

$$= 6 + 8 = 14.$$

EXERCISE. 6·6

Evaluate the following definite integrals as the limits of sums.

1. $\int_0^1 x^3\,dx$; **2.** $\int_0^1 (4x^2+6x+7)\,dx$.

3. $\int_2^4 (2x-1)\,dx$; **4.** $\int_2^3 e^x\,dx$

5. $\int_1^2 (x^2+3x+5)\,dx$ **6.** $\int_3^4 (2x+4)\,dx$;

7. $\int_a^b e^x\,dx$; **8.** $\int_a^b x\,dx$;

9. $\int_a^b x^2\,dx$; **10.** $\int_0^2 \left(e^x + x\right) dx$;

ANSWERS

1. $\frac{1}{4}$; **2.** $\frac{47}{6}$; **3.** 6 ; **4.** $(e^3 - e^2)$.

5. $\frac{71}{6}$; **6.** 11 ; **7.** $e^b - e^a$; **8.** $\frac{b^2-a^2}{2}$;

9. $\frac{b^3-a^3}{3}$; **10.** e^2+1.

6·7. The sum of an infinite series as a definite integral. We already know that

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\left[\sum_{i=1}^{n} hf(a+ih)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{b-a}{n}\right) f\left(a+\frac{b-a}{n}\,i\right)\right]$$

In particular, if $a=0$, $b=1$, then we get

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}\right]$$

Now, suppose that we are given an infinite series $\sum_{n=1}^{\infty} a_n$ which is convergent. If we can write its partial sum s_n as $\left[\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}\right]$ for some suitable function f, then we can obtain the sum of the infinite series $\sum_{n=1}^{\infty} a_n$ as

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}\right]$$

$$= \int_0^1 f(x)\,dx.$$

i.e. The sum of the infinite series can be obtained as a definite integral. It may be noted that while finding the sum of the infinite series as a definite integral, one should be *rather quick* in detecting the function f. Rest is quite simple.

Illustrative Examples

Ex. 1. Evaluate $\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{n}{n^2+i^2}\right]$

Sol. $\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{n}{n^2+i^2}\right]$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1/n}{1+\frac{i^2}{n^2}}\right]$$

(by dividing the numerator and the denominator by n^2)

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}\frac{1}{n}\right]$$

Now, we choose the function f given by

$$f(x) = \frac{1}{1+x^2}.$$

Then, clearly

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{n}{n^2+i^2}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n}\right]$$

$$= \int_0^1 f(x)\,dx$$

$$= \int_0^1 \frac{1}{1+x^2}dx$$

$$= \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

Ex. 2. Evaluate $= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{i^3}{i^4+n^4}\right]$.

Sol.

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{i^3}{n^4+i^4}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{(i^3/n^3)}{1+(i^4/n^4)}\cdot\frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{\left(\frac{i}{n}\right)^3}{1+\left(\frac{i}{n}\right)^4}\cdot\frac{1}{n}\right]$$

$$= \int_0^1 \frac{x^3}{1+x^4}\,dx$$

$$= \frac{1}{4}\int_0^1 \frac{4x^3}{1+x^4}dx$$

$$= \frac{1}{4}\left[\log(1+x^4)\right]_0^1 \quad \left(\because \frac{d}{dx}(1+x^2) = 4x^3\right)$$

$$= \frac{1}{4}\log 2.$$

Note ; One may know the following formulae to evaluate the limit of a sum as a definite integral.

$$(i)\ \lim_{n\to\infty}\left[\sum_{i=0}^{n-1} f\left(\frac{i}{n}\right)\frac{1}{n}\right] = \int_0^1 f(x)\,dx.$$

$$(ii)\ \lim_{n\to\infty}\left[\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}\right] = \int_0^1 f(x)\,dx.$$

$$(iii)\quad \lim_{n\to\infty}\left[\sum_{i=0}^{n+1} f\left(\frac{i}{n}\right)\frac{1}{n}\right]=\int_0^1 f(x)\,dx$$

Ex. 3. Evaluate

$$\lim_{n\to\infty}\left[\sum_{i=0}^{n-1}\sin^2\left(\frac{i\pi}{4n}\right)\frac{1}{n}\right].$$

Sol. Using above formula, we get

$$\lim_{n\to\infty}\left[\sum_{i=0}^{n-1}\sin^2\left(\frac{i\pi}{4n}\right)\frac{1}{n}\right]$$

$$=\int_0^1 \sin^2\left(\frac{\pi}{4}x\right)dx$$

$$=\int_0^1 \frac{1+\cos\left(\frac{\pi}{2}x\right)}{2}dx$$

$$=\frac{1}{2}\left[x+\frac{\sin(\pi/2\,x)}{(\pi/2)}\right]_0^1$$

$$=\frac{1}{2}\left[1+\frac{\sin(\pi/2)}{(\pi/2)}-0\right]$$

$$=\frac{1}{2}\left[1+\frac{2}{\pi}\sin(\pi/2)\right]$$

$$=\left[\frac{1}{2}+\frac{1}{\pi}\right].$$

EXERCISE 6·7

Using the definition of the limit of a sum as a definite integral prove the following.

1. $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]=\log 2,$

2. $\lim_{n\to\infty}\left[\sum_{i=0}^{n-1}\frac{1}{n+i}\right]=\log 2.$

3. $\lim_{n\to\infty}\left[\sum_{i=0}^{n-1}\frac{n^2}{(n+i)^3}\right]=\frac{3}{8}.$

4. $\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{n^3}{(n+i)^4}\right]=\frac{7}{24}.$

5. $\lim_{n\to\infty}\left[\frac{n^4}{(n+1)^5}+\frac{n^4}{(n+2)}+\cdots+\frac{n^4}{(n+n)^5}\right]$
$=\frac{15}{64}$.

6. $\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{\sqrt{n^2-i^2}}\right]=\frac{\pi}{2}$

7. $\lim_{n\to\infty}\left[\frac{e^{1/2}+e^{2/n}+\cdots e^{n/n}}{n}\right]=e-1$

8. $\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{\pi^2 i\sin\left(\frac{i\pi}{n}\right)}{n^2}\right]=\pi$.

9. $\lim_{n\to\infty}\left[\frac{1}{n^3+1}+\frac{4}{n^3+2^3}+\frac{9}{n^3+3^3}+\cdots+\frac{1}{2n}\right]$
$=\frac{1}{3}\log 2$.

10. $\lim_{n\to\infty}\left[\frac{1}{n^5+1}+\frac{16}{n^5+2^5}+\frac{81}{n^5+3^5}+\cdots+\frac{1}{2n}\right]$
$=\frac{1}{5}\log 2$.

11. $\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{n^{1/2}}{\sqrt{(n+8i)^3}}\right]=\frac{1}{6}$.

6·8. Area under a curve as a definite Integral

We shall now obtain a geometrical interpretation of the definite integral.

Th. If the function f is continuous on the closed interval $[a, b]$ and $f(x)\geqslant 0$ for all x in $[a, b]$ then area bounded by the curve $y=f(x)$, the x—axis and the ordinates at $x=a$ and $x=b$ is given by $\int_a^b f(x)\,dx$.

Proof. Suppose $f(x)$ is increasing in $[a, b]$. Let AB denote the graph of $y=f(x)$ from $x=a$ to $x=b$ as shown in fig. 6·4.

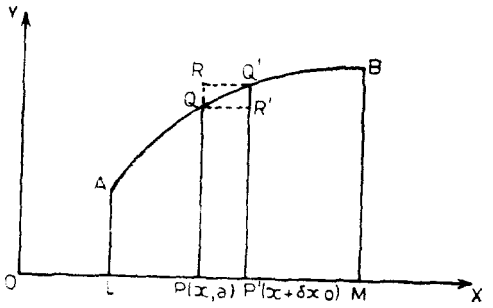

Fig. 6·4.

Draw AL, BM perpendicular to OX, meeting it in $L\ (a, 0)$ and $M\ (b, 0)$ respectively. Let $P\ (x, 0)$ be a point between L and M on the axis of x. Let the ordinate at P meet the graph in Q.

Since Q is a point on the graph whose abscissa is x, the ordinate of Q is $f(x)$ *i.e.*, $PQ = f(x)$.

Now let $F(x) =$ area $ALPQ$. This area is determined by the position of P between L and M *i.e.*, by the abscissa of P *viz.*, x. Hence it is a function of x. That is why we have used the symbol $F(x)$ for this area. Let Q' be a neighbouring point on the graph *i.e.* with co-ordinates $\{(x+\delta x), f(x+\delta x)\}$. Draw $Q'P'$ perpendicular to OX meeting it in P'. Then $P'Q' = f(x+\delta x)$.

Draw QR' perpendicular to $Q'P'$ meeting it in R' and $Q'R$ perpendicular to PQ (produced) meeting it in R. Then from the figure we have

$$P'R' = PQ = f(x)$$
$$PR = P'Q' = f(x+\delta x)$$
$$OP = x,\ OP' = x+\delta x$$
$$\therefore\ PP' = OP' - OP = \delta x$$

Now area $ALPQ = F(x)$

area $ALP'Q' = F(x+\delta x)$

$\therefore$ The area bounded by the curve QQ', the x-axis and the ordinates PQ and $P'Q'$ *i.e.*, the area $QPP'Q'$ is given by

$$\text{area QPP'Q'} = \text{area } ALP'Q' - \text{area } ALPQ$$
$$= F(x+\delta x) - F(x)$$

Also area of the rectangle $PP'R'Q = PQ \times PP' = f(x) \cdot \delta x$ and area of the rectangle $PP'Q'R = P'Q'XPP' = f(x+\delta x) \cdot \delta x$. It will be seen from the figure that, area of rect $PP'R'Q <$ area $QPP'Q <$ area of rect $PP'Q'R$, *i.e.*, $f(x) \cdot \delta x < F(x+\delta x) - F(x) < f(x+\delta x) \cdot \delta x$

$\therefore$ we have

$$f(x) < \frac{F(x+\delta x) - F(x)}{\delta x} < f(x+\delta x)$$

Taking now limits as $\delta x \to 0$ since

$$\lim_{\delta x \to 0} \frac{F(x+\delta x) - F(x)}{\delta x} = F'(x)$$

and $\lim_{\delta x \to 0} f(x+\delta x) = f(x)$ as f is a continuous function we get

$$f(x) \leqslant F'(x) \leqslant f(x)$$

i.e. $F'(x) = f(x)$

$\therefore$ $F(x)$ is a primitive of $f(x)$.

$$\therefore \int_a^b f(x)\,dx = [F(x)]_a^b$$
$$= F(b) - F(a)$$

Now by the definition of $F(x)$

$F(b)$ = area $ALMB$

and $F(a) = 0$

$$\therefore \int_a^b f(x)\,dx = \text{area } ALMB$$

Hence geometrically $\int_a^b f(x)\,dx$ represents the area of the region bounded by the curve $y = f(x)$, the axis of x and the ordinates at $x = a$ and $x = b$.

The function f is assumed to be continuous. We have drawn the graph of an increasing function. The same result is obtained when the function is decreasing or sometimes increasing and sometimes decreasing. The only restriction is that the graph must not cross the axis of x between $x = a$ and $x = b$. When it crosses the axis of x say once we consider the two areas to the left and the right of the point of intersection separately.

Sign of the Area

Consider the area A of the region bounded by the curve $y = \sin x$, the x-axis and the lines $x = 0$, $x = 2\pi$. As seen above, this area $A = \int_0^{2\pi} \sin x\,dx$.

$$= [-\cos x]_0^{2\pi}$$
$$= -\cos 2\pi + \cos 0$$
$$= 1 - 1 = 0$$

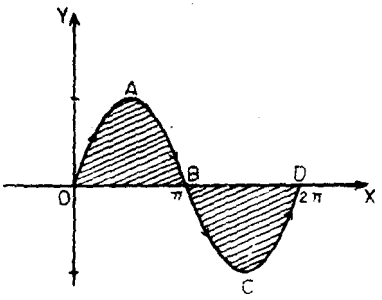

Fig. 6·5.

From the adjoining figure, we see that there is area of the *region*, but by the above calculations, the area has come out to be zero. Let us try to know the reason for this apparent absurdity.

The whole region R is composed of two subregions $OABO$ and $BCDB$. Out of this the region $OABO$ is above the x-axis and hence its area is positive. The region $BCDB$ is below the x-axis and its area is $\int_{\pi}^{2\pi} y\,dx$. In this region $y = f(x)$ is negative and hence,

$S_n = \sum_{r=1}^{n} f(t_r)\,\delta r$ is also negative.

Therefore the area of this region is negative.

By symmetry the area of these two subregions are equal in magnitude, but they are opposite in sign.

$$\text{Area of } OABO = \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi}$$

$$= -\cos \pi + \cos 0 = 1 + 1 = 2$$

$$\text{Area of } BCDB = \int_{\pi}^{2\pi} \sin x\,dx = [-\cos x]_{\pi}^{2\pi}$$

$$= -\cos 2\pi + \cos\pi = -1 - 1 = -2$$

The magnitude of the area of the region R under the curve $y = \sin x$, from $x = 0$ to $x = 2\pi$ will then be

$$= \int_0^{\pi} \sin x\,dx + \left| \int_{\pi}^{2\pi} \sin x\,dx \right| = 2 + 2 = 4$$

Sketching of a curve

In order to find the area under a given curve it is necessary to draw a rough sketch of the curve. For this purpose the following points will be useful.

(*i*) **Symmetry**

Consider the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as shown in the figure. We can at once say that the area of the ellipse is four times the area of the region in the first quadrant (shaded in the figure).

This happens because the ellipse is symmetrical about both the axes. Thus the symmetry of the curve helps us in calculating the area under that curve.

We shall find the area of the region $OABO$. The equation of the curve in this region is $y = \frac{b}{a}\sqrt{a^2-x^2}$.

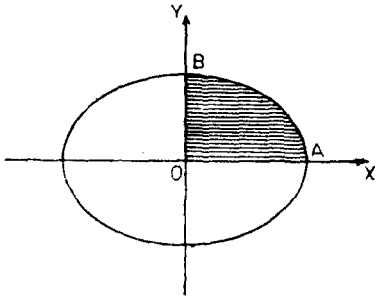

Fig. 6·6.

The limits of integration will be the x-coordinates of O (origin) and the point A. These are obviously 0 and a.

$$\therefore \quad \text{Area of region } OABO = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\,\frac{a^2}{2}\,(\sin^{-1}1 - \sin^{-1}0)$$

$$= \frac{\pi}{4}\,ab$$

$\therefore$ The area of the ellipse $= 4$ (area of region $OABO$)

$$= 4 \cdot \frac{\pi}{4}\,ab = \pi\,ab$$

We have already known about symmetry in coordinate geometry and we shall be making use of it in finding the area of region which is symmetrical. By the same method show that, the area of the circle $x^2+y^2 = a^2$ is πa^2

(*ii*) Points of intersection with the coordinate areas

We know that the y-coordinate of a point on the x-axis is zero. Hence in order to find the x-coordinates of the points of intersection of the curve with the x-axis, put $y = 0$ in the equation of the curve. The resulting equation will be an equation in x and its roots, will then be the x-coordinates of these points of intersection.

Similarly putting $x = 0$, in the equation of the curve we shall get an equation in y. The roots of this equation will be the y-coordinates of the points of intersection of the curve with the x-axis, *e.g.* Let the equation of the curve be $y = x^2 - 2x - 3$.

Putting $y = 0$, we get $x^2 - 2x - 3 = 0$

$\therefore \quad (x+1)(x-3) = 0$

$\therefore$ The roots of this equation are $x_1 = -1$ and $x_2 = 3$.

These are the x-coordinates of the points in which the curve intersects the x-axis. Their coordinates are $(-1, 0)$ and $(3, 0)$.

Similarly, putting $x = 0$, we get $y = -3$.

This is the y-coordinate of the point of intersection of the curve with y-axis. This point is $(0, -3)$.

Let us try to draw a rough sketch of this curve.

The equation of the curve can be written as, $y = (x^2 - 2x + 1) - 4$ *i.e.*, $y + 4 = (x-1)^2$. Put $Y = y + 4$ and $X = x - 1$. Then the equation becomes $Y = X^2$. We know that this is the equation of a parabola. Thus the given equation represents a parabola, which cuts the x-axis in $A\ (-1, 0)$; $B\ (3, 0)$ and the y-axis in $C\ (0, -3)$.

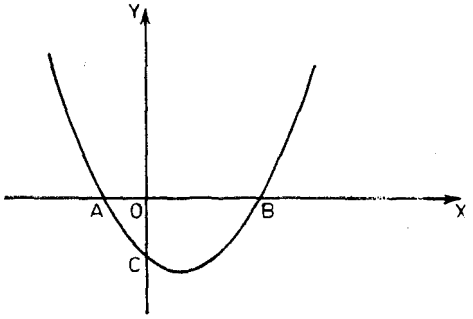

Fig. 6·7.

The formulae of transformation are

$$x = X + 1 \text{ and } y = Y - 4$$

Thus the new origin is $(1, -4)$. But this is the vertex of the parabola.

With this much information, we can draw a rough sketch of parabola.

The following illustrative examples will make the above ideas still more clear.

Illustrative Examples

Ex. 1. Find the area of the portion of the parabola $y^2 = 4ax$ from its vertex to its latus rectum.

Sol. From the sketch of parabola the required region is $OLSL'O$ where $S\ (a, 0)$, is the focus and LSL' is the latus rectum.

The parobola is symmetrical about the x-axis.

$\therefore$ Area of region $OLSL'O$ = 2 area of region $OLSAO$.

In this region the equation of the curve is $y = 2\sqrt{ax}$. The limits of integration are the x-coordinates of O (origin) and S. These are obviously 0 and a.

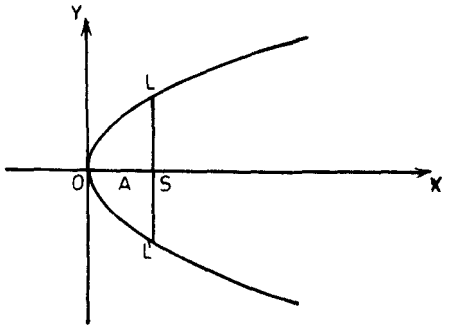

Fig. 6·8.

$\therefore$ Area of region $OLSL'O$

$$= 2\int_0^a \sqrt{ax}\,dx = 4\sqrt{a}\int_0^a x^{\frac{1}{2}}dx$$

$$= 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a = \frac{8\sqrt{a}}{3}(a^{3/2}-0)$$

$$= \frac{8a^2}{3}$$

Ex. 2. Find the area of the region enclosed by the parabola $y = x^2-6x+11$ and the line $y-2x+1=0$.

Sol. Let the parabola and the line intersect each other in $P\ (x_1, y_1)$ and $Q\ (x_2, y_2)$. The x-coordinates of P and Q are given by

$y = x^2-6x+11 = 2x-1$

i.e., $x^2 - 8x+12 = 0$

i.e., $(x-2)\ (x-6) = 0$

$\therefore$ $x_1 = 2$ and $x_2 = 6$

From the fig. we want to find the area of the region $PAQBP$. Also, area of $PAQBP$ = Area of $PP'Q'QBP$—Area $PP'Q'QAP$ ...(1)

Region $PP'Q'QBP$ is bounded by the line $y = 2x-1$ the x-axis and the lines $x = 2$ and $x = 6$.

$\therefore$ Area of $PP'Q'QBP$

$$= \int_2^6 (2x-1)\, dx$$

$$= \left[x^2-x\right]_2^6 = (36-6)-(4-2) = 28$$

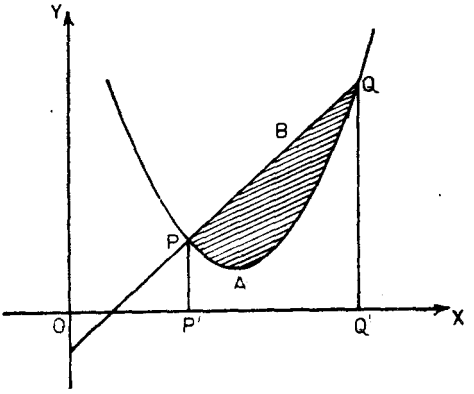

Fig. 6·9.

Region $PP'Q'QAP$ is bounded by the parabola $y = x^2-6x+1$, the x-axis and the lines $x = 2$ and $x = 6$.

$$\therefore \text{ Area of } PP'Q'QAP = \int_2^6 (x^2-6x+11)\, dx$$

$$= \left[\frac{x^3}{3} - 3x^2+11x\right]_2^6$$

$$= (72-108+66)-(\tfrac{8}{3}-12+22)$$

$$= 30-\tfrac{38}{3} = \tfrac{52}{3}$$

$\therefore$ (1) gives

Area of $PAQ'BP = 28-\tfrac{52}{3} = \tfrac{2}{3}$

Ex. 3. Find the area of the region common to the circle $x^2+y^2 = 8$ and the parabola $y^2 = 2x$.

Sol. Let the circle and the parabola intersect each other in $A(x_1, y_1)$ and $B(x_2, y_2)$. (By symmetry A and B have the same x-coordinate and it is positive.

From the equations of the circle and the parabola equating the values of y^2,

We get,

$$y^2 = 8-x^2 = 2x$$

i.e., $x^2+2x-8 = 0$ *i.e.*, $(x+4)(x-2) = 0$

$\therefore\ x = -4$ and $x = 2$

$x = -4$ is rejected, since this will make y^2 negative for the parabola and the circle.

$\therefore\ x = 2$ and x-coordinate of A is 2. Also x-coordinate of C is $2\sqrt{2}$.

We want to find the area of the region $OACBO$. By symmetry, this area is twice the area of the region $OACMO$ (in the first quadrant).

$\therefore$ Area of $OACBO$

$= 2$ area of $OACMO$

$= 2$(area of $OAMO$ + area of $ACMA$)

$$= \left[\int_0^2 \sqrt{2x}\,dx + \int_2^{2\sqrt{2}} \sqrt{8-x^2}\,dx\right.$$

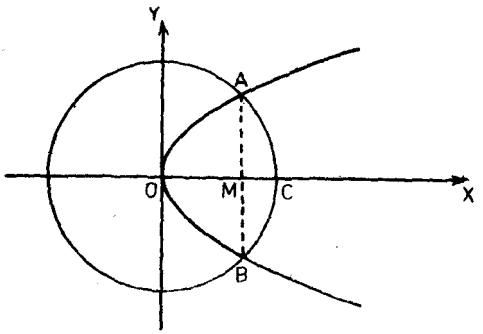

Fig. 6·10

$$= \sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^2 + 2\left[\frac{x}{2}\sqrt{8-x^2} + \frac{8}{2}\sin^{-1}\frac{x}{2\sqrt{2}}\right]_2^{2\sqrt{2}}$$

$$= \frac{4\sqrt{2}}{3} \times 2\sqrt{2} + 2\left[4\sin^{-1}1 - 2 - 4\sin^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= \frac{16}{3} + 2(2\pi - 2 - \pi) = \frac{16}{3} + 2(\pi - 2)$$

Ex. 4. Find the area of the region common to the parabolas $y^2 = 4ax$ and $x^2 = 4ay$

Sol. First we find the x-coordinates of the points of intersection of the parabolas.

From the equations of the parabolas, we have,

$$y^2 = 4ax \text{ and } y^2 = \frac{x^4}{16a^2}$$

$$\therefore \quad \frac{x^4}{16a^3} = 4ax \qquad \therefore \quad x^4 - 64\,a^3x = 0$$

$$\therefore \quad x\,(x^2 - 64a^3) = 0 \qquad \therefore \quad x = 0 \text{ and } x = 4a$$

These are the limits of integration.

Required area =

= area of $OAPBO$

= area of $OAPP'O$ — area of $OBPP'O$

$$= \int_0^{4a} 2\sqrt{ax}\,dx - \int_0^{4a} \frac{x^2}{4a}\,dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

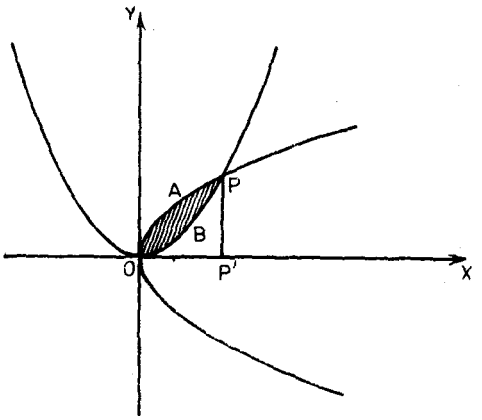

Fig. 6·11.

EXERCISE 6·8

1. In each of the following problems, find the area bounded by the given curve, the x-axis and the given lines

(*i*) $y = x^2 + 3$, $x = 0$, $x = 3$

(*ii*) $y = x^2-3x+4$, $x = 0$, $x = 2$.

(*iii*) $y = 2-x^2$, $x = -1$, $x = 1$.

(*iv*) $y = 3-2x-x^2$, $x = -1$, $x = 3$.

(*v*) $y^2 = 4x$, $x = 2$, $x = 4$.

2. In each of the following problems, find the area of the region enclosed between the given curve and the line

(*i*) $y = x^2-5x+15$, $y = 3x+3$.

(*ii*) $x^2+4y = 4$, the x-axis.

(*iii*) $y = 2-x^2$, $y+x = 0$.

(*iv*) $y^2 = 8x$, its latus rectum.

(*v*) $y^2 = 8x$, $2x-3y+8 = 0$.

(*vi*) $ay = 3(a^2-x^2)$, x-axis.

3. Show that the area of the region bounded by the curve $y = x^2-2x$, the x-axis and the line $y = 2x$ is $\frac{49}{3}$ but the area of the region between the curve and the line $y = 2x$ is $\frac{125}{6}$.

4. Find the area of the region of the parabola $x^2+4y = 4$ in the first two quadrants.

5. Find the area of the region in the first quadrant enclosed by the x-axis, the line $x = y\sqrt{2}$ and the ellipse $x^2+2y^2 = 4$.

6. Show that the area of the region in the first quadrant, enclosed by the x-axis, the line $x = y\sqrt{3}$ and the circle $x^2+y^2=4$ is $\frac{\pi}{3}$.

7. Find the area of the region common to the following curves :

(*i*) $y^2 = 4x$, $x^2 = 4$.

(*ii*) $y = 5x^2$, $2x^2-y+9 = 0$.

(*iii*) $x^2+y^2 = 12$, $y^2 = 4x$.

(*iv*) $x^2+3y^2 = 3$, $3x^2+y^2 = 3$.

(*v*) $y^2 = 16x$, $x^2+y^2 = 36$.

(*vi*) $y^2 = 3x$, $y^2 = -3x$, $x^2+y^2 = 4$.

3. Find the area of the region above the x-axis enclosed between the curves $y^2 = 2ax-x^2$ and $y^2 = ax$.

ANSWERS

1. (*i*) 18 ; (*ii*) $\frac{14}{3}$; (*iii*) $\frac{10}{3}$; (*iv*) Hint : The area from -1 to 1 is positive. Ans. 16, (*v*) $\frac{8}{3}(4-\sqrt{2})$

2. (*i*) $\frac{32}{3}$; (*ii*) $\frac{8}{3}$; (*iii*) $\frac{9}{2}$; (*iv*) $\frac{32}{3}$; (*v*) $\frac{4}{3}$; (*vi*) $4a^2$.

4. $\frac{8}{3}$.

5. $\frac{\pi}{2\sqrt{2}}$.

7. (*i*) $\frac{16}{3}$; (*ii*) $12\sqrt{3}$; (*iii*) $2\left(2\sqrt{2}+3\pi-\sin^{-1}\frac{1}{\sqrt{3}}\right)$; (*iv*) $\frac{2\pi\sqrt{3}}{3}$; (*v*) $18\pi+\frac{8\sqrt{2}}{3}-36\sin^{-1}\frac{1}{3}$; (*iv*) $\frac{2}{3}\left(2\pi-\sqrt{3}\right)$

8. $\left(\frac{2}{3}+\frac{\pi}{4}\right)a^2$.

6·8. Volume of Revolution.

Let AB be an arc of a curve and XY be a line not intersecting the arc AB.

Draw segments AA' BB' perpendicular to the line XY.

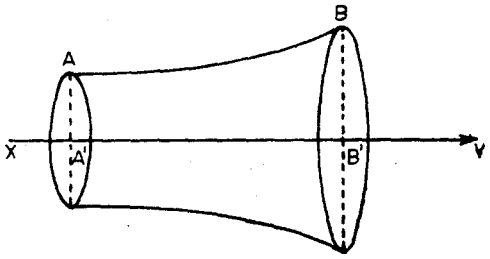

Fig. 6·12

If the region AA' $B'B$ is revolved completely about the line XY, the solid, so obtained is called the *Solid of revolution.*

The line XY about which the plane region AA' $B'B$ is revolved is called the *axis of revolution.*

e. g. (1) If a semicircle is revolved about its diameter, the solid so obtained is a sphere.

(2) If a right-angled triangle is revolved about a line containing an arm of a rightangle, the solid of the revolution so obtained is a cone.

(3) If a rectangle is revolved about a line containing one of its sides, the solid of revolution so obtained is a cylinder.

We shall assume that the volume of a cylinder is equal to the product of the area of its base and its height, and obtain the volume of the solid of revolution in the following theorem.

Theorem. If the function f is continuous on the closed interval $[a, b]$ and $f(x)\geqslant 0$ for all x in $[a, b]$, then the volume of the solid obtained by revolving about the x—axis, the region bounded

by the curve $y=f(x)$ the x—axis and the lines $x=a$ and $x=b$ is given by $\int_a^b \pi y^2\, dx$.

Proof. Let AB be the arc of the curve $y=f(x)$ from $x=a$ to $x=b$.

Draw segments AA' BB' perpendicular to the x—axis. If the region AA' $B'B$ is revolved about the x=axis, we get a solid of revolution. We want to find the volume of this solid of revolution.

Let $P\ (x, f(x))$ and $Q\ (x+\delta x, f(x+\delta x))$ be two points on the arc AB. Draw segments PP' and QQ' perpendicular to the x—axis.

Then $PP'=f(x)$; $QQ'=f(x+\delta x)$; $P'Q'=\delta x$.

Let $V(x)$ denote the volume of the solid AA_1P_1PA, and δV the volume of the strip PP_1Q_1QP, (Fig. 6·13).

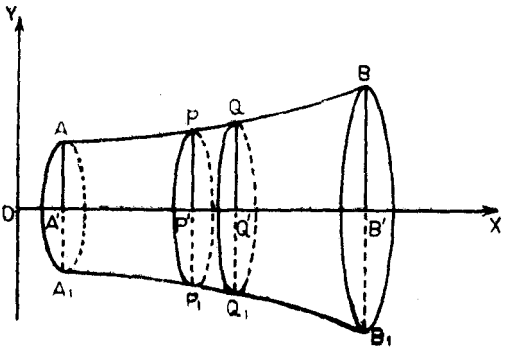

Fig. 6·13

The adjoining figure shows the cross-section of the solid by the XY—Plane. (Fig. 6·14)

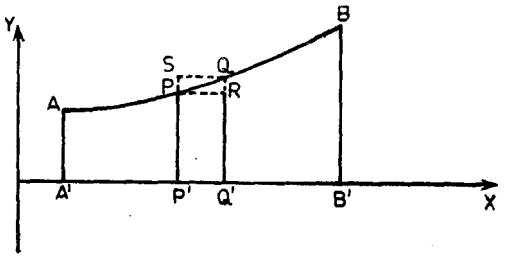

Fig. 6·14

Area of rect. $PP'\ Q'\ R$ <area of $PP'\ Q'\ QP$<area of rect. $SP'\ Q'\ Q$.

The same inequalities hold for the volumes of the solids obtained by revolving about the x—axis, the above regions. ...(1)

When the rectangular region $PP'Q'R$ is revolved about the x—axis the solid of revolution is a right circular cylinder of base radius $PP'=f(x)$ and height $P'Q'=\delta x$.

$\therefore$ Vol. of this cylinder $=\pi\ (PP')^2\ (P'Q')$

$$=\pi[f(x)]^2\delta x$$

when the rectangular region $SP'Q'Q$ is revolved about the x—axis, the solid of revolution is a right circular cylinder of base radius $QQ'=f(x+\delta x)$ and height $P'Q'=\delta x$.

$\therefore$ Vol. of this cylinder $=\pi\ (QQ')^2\ (P'Q')$

$$\pi=[f(x+\delta x)]^2\ \delta x$$

$\therefore$ From (1) we get,

$$\pi\ [f(x)]^2\ \delta x<\delta V<\pi[f(x+\delta x)]^2\ \delta x$$

$$\therefore\quad \pi\ [f(x)]^2<\frac{\delta V}{\delta x}<\pi\ [f(x+\delta x)]^2$$

$$\therefore\quad \lim_{\delta x\to 0}\ \pi\ [f(x)]^2\leqslant \lim_{\delta x\to 0}\ \frac{\delta V}{\delta x}\leqslant \lim_{\delta x\to 0}\ \pi\ [f(x+\delta x]]^2 \qquad ...(2)$$

Now f is continuous at x.

$$\therefore\quad \lim_{\delta x\to 0} f(x+\delta x)=f(x)$$

$\therefore$ (2) gives,

$$\lim_{\delta x\to 0}\ \frac{\delta V}{\delta x}=\pi\ [f\ (x)]^2$$

$$\therefore\quad \frac{dV}{dx}=\pi y^2$$

Integrating this relation between the limits a and b we get,

$V\int_v^b=\int_a^b \pi y^2\ dx$, which proves the result.

Illustrative Examples.

Ex. 1. Find the volume of the sphere of radius a.

Sol. A sphere of radius a can be considered as a solid obtained by revolving about the x—axis the semicircle $x^2+y^2=a^2$ above the x—axis.

$$\therefore\quad V=\int_{-a}^{a} \pi y^2\ dx.$$

$$=\int_{a-}^{a} \pi\ (a^2-x^2)\ dx=2\pi\int_0^a\ (a^2-x^2)\ dx.$$

$$=2\pi\left[a^2x-\frac{x^3}{3}\right]_0^a=\frac{4}{3}\ \pi a^3$$

Ex. 2. A plane is drawn parallel to the base of a hemisphere of radius a at a distance of $a/3$ from the centre of the base. Find

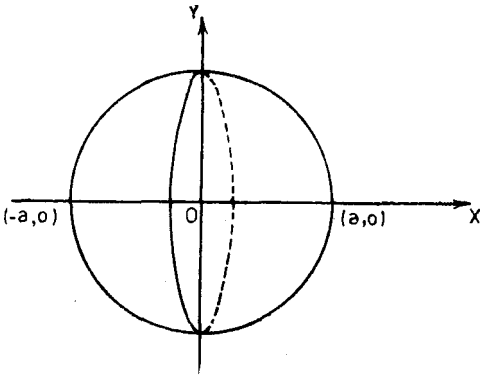

Fig. 6·15

the ratio of the volumes of the two portions in which the plane divides the hemisphere.

Sol. Let ABC be the hemisphere with O as the centre of the base. Choose the coordinate axes as shown in the adjoining fig. Let DE be the plane at a distance $\frac{a}{3}$ from O and parallel to the base.

Then the two portions are $BCED$ and ADE. These can be considered as solids obtained by revolving about the x—axis, the plane regions $BOMD$ and DMA respectively.

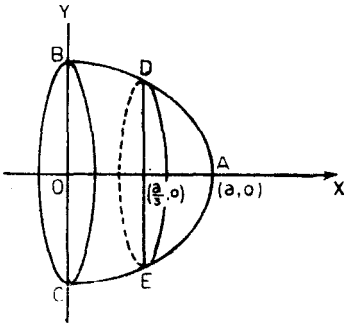

Fig. 6·16

The equation of the circle in the xy—plane is $x^2+y^2=a^2$ *i.e.* $y^2=a^2-x^2$.

$$\therefore \quad \text{Vol. of } BCED=\int_0^{a/3}\pi\ y^2dx$$

$$= \pi \int_0^{a/3} (a^2 - x^2)\, dx.$$

$$= \pi \left[a^2 x - \frac{x^3}{3} \right]_0^{a/3} = \pi \left(\frac{a^3}{3} - \frac{a^3}{81} \right)$$

$$= \frac{26\, a^3}{81} \qquad \ldots(1)$$

$$\text{Vol. of } ADE = \int_{a/3}^{a} \pi y^2\, dx = \pi \int_{a/3}^{a} (a^2 - x^2)\, dx$$

$$= \pi \left[a^2 x - \frac{x^3}{3} \right]_{a/3}^{a} = \pi \left[\left(a^3 - \frac{a^3}{3} \right) - \left(\frac{a^3}{3} - \frac{a^3}{81} \right) \right]$$

$$= \frac{2\pi\, a^3}{3} - \frac{26 a^3}{81} = \frac{28 a^3}{81} \qquad \ldots(2)$$

$$\frac{\text{Vol. of } BCED}{\text{Vol. of } ADE} = \frac{26}{28} = \frac{13}{14}$$

Ex. 3. The parabola $y^2 = 4ax$ is revolved about the x—**axis.** Find the volume of the solid of revolution between the vertex and the plane perpendicular to the axis through the focus.

Sol. Let $S(a, 0)$ be the focus.

$\therefore$ Required volume

$$= \int_0^a \pi y^2\, dx$$

$$= \pi \int_0^a 4ax\, dx$$

$$= 4\, a\pi \left[\frac{x^2}{3} \right]_0^a$$

$$= 2\pi a^3.$$

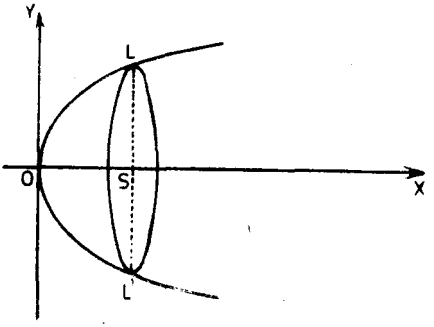

Fig. 6·17

Ex. 4. The rectangular hyperbola $xy = 1$ is revolved about the x—**axis.** Find the volume of the solid of revolution included between $x = 1$ and $x = 2$.

Sol. The equation of the rectangular hyperbola is $xy=1$ *i.e.* $y=\frac{1}{x}$.

$$\therefore \text{ Required volume} = \int_1^2 \pi y^2\, dx$$

$$= \pi \int_1^2 \frac{dx}{x^2} = \pi \left[\frac{-1}{x}\right]_1^2$$

$$= \pi\left(-\frac{1}{2}+1\right) = \frac{\pi}{2}$$

Ex. 5. The straight line $x-y+3=0$ cuts the parabola $y^2=16x$ in the points A and B. Find the volume of the solid obtained by revolving about the x—axis the region enclosed between the parabola and the chord AB.

Sol. Eliminating y between the equations of parabola and the line, we get,

$$(x+3)^2=16x$$

i.e. $x^2-10x+9=0$

$\therefore$ $x=1$ or $x=9$

$\therefore$ A is (1, 4) and B is (9, 12).

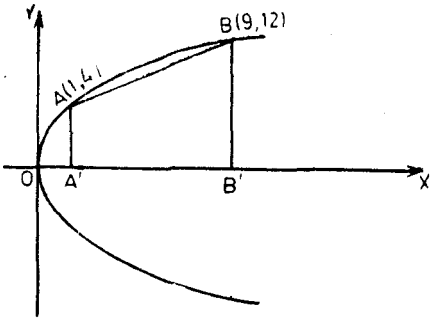

Fig. 6·18

From the Fig. it is clear that the required.

Volume $V=V_1-V_2$ where

V_1=Vol. of the solid obtained by revolving the region below the arc AB about the x—axis.

$$= \int_1^9 \pi y^2\, dx \qquad (\text{where } y^2=16x)$$

$$= \int_1^9 16x\, dx = 16\pi \left[\frac{x^2}{2}\right]_1^9$$

$$= 8\pi\,(81-1) = 640\pi$$

and V_2=Vol. of the solid obtained by revolving the region below the chord AB about the x—axis

$$= \int_1^9 \pi y^2 \, dx \qquad (\text{where } y = x+3)$$

$$= \pi \int_1^9 (x+3)^2 dx = \pi \left[\frac{(x+3)^3}{3} \right]_1^9$$

$$= \pi \left(\frac{1728}{3} - \frac{64}{3} \right) = \frac{1664}{3} \pi$$

$$\therefore \quad V = V_1 - V_2 = 640 \ \pi - \frac{1664}{3} \pi$$

$$= \frac{256}{3} \pi$$

EXERCISE 6·9.

1. Obtain the volume of a right circular cone of

   (*i*) height h and semivertical angle α.

   (*ii*) height h and base radius r.

2. Find the volume of the frustum of a cone with radii of the two faces r_1 and r_2 and height h.

3. A plane is drawn parallel to the base of a hemisphere of radius a at a distance $\frac{a}{2}$ from the centre of its base show that the plane divides the volume of the hemisphere in the ratio 11 : 5.

4. A sphere of radius a is cut into two parts by means of a plane at O distance $a-h$, $0<h<a$, from the centre of the sphere. Find the volume of the smaller part of the sphere.

5. Show that the volume of a cap of height h_2 of a sphere of radius r is $\frac{\pi h^2}{3}(3r-h)$.

6. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated about the x—axis. Show that the volume of the solid so obtained is $\frac{4}{3} \pi ab^2$

7. Find the volume of the solid obtained by rotating the region enclosed by the ellipse $x^2+4y^2=4$ about the x-axis.

8. The parabola $y^2=16x$ is revolved about the x-axis. The solid so obtained is cut by a plane perpendicular to the axis and passing through the focus. Find the volume of the solid between the vertex and the plane.

9. One branch of the rectangular hyperbola $x^2-y^2=a^2$ is revolved about the x-axis to form a solid of revolution. Prove that the volume of

the portion of the solid of height a measured from the vertex is equal to the volume of the sphere of radius a.

10. Show that volume of the solid generated by revolving about the x-axis, the region in the first quadrant enclosed by the parabola $y^2=4x$ and the line $x=2$, $x=4$ is 18π.

11. The part of the parabola $x^2+4y=4$ in the first two quadrants revolves about the x-axis. Find the volume of the solid generated.

12. Show that volume of the solid obtained by revolving about the x-axis the region enclosed by the parabola $ay=3(a^2-x^2)$ and the x-axis, is $\frac{48\pi a^3}{5}$.

13. AOB is the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $OA=a$ and $OB=b$. Show that the volume of the solid obtained by revolving about the x-axis, the region enclosed between the chord AB and the arc AB is $\frac{\pi}{3}ab^2$. Also show that the area of this plane region is $\frac{1}{4}(\pi-2)ab$.

14. Show that the area of the region in the first quadrant, enclosed by the x-axis, the line $x=y\sqrt{3}$ and the circle $x^2+y^2=4$ is $\frac{\pi}{3}$

Show further that if this region is revolved about the x-axis, the volume of the solid so generated is $\frac{8\pi}{3}(2-\sqrt{3})$.

15. The region in the first quadrant bounded by the x-axis, the parabola $y^2=4x$ and the circle $x^2+y^2=12$ revolves about the x-axis. Find the volume of the solid of revolution.

16. A straight line $2x-3y+8+0$ cuts the parabola $y^2=8x$ in the points A and B. Find the volume of the solid generated by revolving the region, enclosed between the parabola and the chord AB, about the x-axis.

17. Find the volume of the solid obtained by revolving about the x axis the region enclosed between the parabola $y=x^2+1$ and the line

$$y=2x+1$$

18. The curve $y=\sin x$, $0\leq x\leq\pi$ is rotated about the x-axis. Find the volume of the solid generated.

19. Find the volume of the solid generated by revolving about the x-axis the region bounded by the parabolas

$$x^2=2y \text{ and } x^2=12-4y.$$

20. The solid obtained by revolving the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

about the x-axis ($a>b$) is cut by a plane through one of its foci and perpendicular to the major axis. Show that the ratio of the volumes of the two part is $2+3e+e^2 : 2-3e-e^2$, where e is the eccentricity of the ellipse.

ANSWERS

1. (*i*) $\frac{1}{3}\pi h^3 \tan^2\alpha$; (*ii*) $\frac{1}{3}\pi r^2 h$.

2. $\frac{1}{3}\pi h\,(r_1^2+r_1r_2+r_2^2)$; **4.** $\frac{\pi}{3}h^2\,(3r-h)$;

7. $\frac{8\pi}{3}$; **8.** $128\,\pi$; **11.** $\frac{32\,\pi}{15}$;

15. $8\pi\left(2\sqrt{3}-\frac{5\pi}{3}\right)$; **16.** 16π ;

17. $\frac{104\,\pi}{15}$; **18.** $\frac{\pi^2}{2}$; **20.** $\frac{128\,\pi}{5}$.

7

Differential Equations

7·1. What is a differential equation

The student is familiar with algebraic equations such as

$$3x + 5 = 0$$

$$3x^2 + 7x - 10 = 0$$

in which x is an unknown quantity to be determined in such a way that the given equation is satisfied. In calculus, many times, a variable y is an unknown function of another variable x. Their relationship is given in terms of the derivatives of y w.r. t x. This leads to equations involving the derivatives of y w.r. t x along with x and y. Such equations are called *differential equations.* A differential equation thus is an equation involving differential coefficients. The following are some illustrations.

$$y\frac{dy}{dz} + y + 2x = 0 \qquad (i)$$

$$\left(\frac{dy}{dx}\right)^2 + (1-x^2)y = 0 \qquad (ii)$$

$$\frac{d^2y}{dx^2} + a^2y = 0 \qquad (iii)$$

$$\left(\frac{d^2y}{dx^2}\right)^2 + y\frac{dy}{dx} + x^2 = 0 \qquad (iv)$$

7·2. Order and degree of a differential equation.

The *order* of a differential equation is the order of the highest order differential coefficient involved in it. Thus in the above (*i*) and (*ii*) are of the first order while (*iii*) and (*iv*) are of order 2.

The *degree* of a differential equation is the degree of the highest

order differential coefficient involved in it. In the above illustrations (*i*) and (*iii*) are of degree 1 while (*ii*) and (*iv*) are of degree two.

7·3. Formation of differential equation

Differential equations generally arise from geometrical or physical problems. We shall first consider how differential equations arise from geometry.

Consider the equation,

$$y=2x+c \qquad (i)$$

This equation represents a straight line with slope 2 and y-intercept c. For different values of c it represents different lines. Further any line with slope 2 will be represented by the above equation, for an appropriate value of c. Hence this equation represents all straight lines with slope 2. The equation thus represents a family of lines. The constant c which can be assigned any arbitrary value, is called a parameter.

Now from (I) we obtain by differentiation,

$$\frac{dy}{dx} = 2 \qquad (ii)$$

Equation (*ii*) is true for every line of the family represented by equation (*i*). It simply states the fact that the slope of every line in the family $\left(\text{represented by } \frac{dy}{dx}\right)$ is 2. Observe that the constant c has disappeared in the process of differentiation. We have obtained an equation involving a differential coefficient *i.e.* a differential equation. This differential equation is satisfied by the equation of every line in the family represented by equation (*i*). It is called the differential equation of the family and is obtained by eliminating the parameter involved in the equation of the family by differentiation,

Similarly the equation

$$x^2+y^2=c^2 \qquad (iii)$$

in which c is a parameter represents the family of circles with centre at the origin. From (*iii*) we obtain

$$x + y\,\frac{dy}{dx} = 0 \qquad (iv)$$

by eliminating the parameter by differentiation. This is the differential equation of the family of circles represented by equation (*iii*).

Now consider the equation,

$$y=a \sin (bx+c) \qquad (v)$$

in which a and c are parameters and b is an absolute constant. This equation represents a family of sine curves. We shall obtain the

differential equation of this family by differentiation. From (v) we obtain,

$$\frac{dy}{dx} = a\,b\,\cos\,(bx + c) \qquad (vi)$$

Differentiating again we obtain,

$$\frac{d^2y}{dx^2} = -ab^2 \sin\,(bx+c) \qquad (vii)$$

Since $a \sin\,(bx+c)=y$, (*vii*) reduces to,

$$\frac{d^2y}{dx^2} = -b^2y$$

OR

$$\frac{d^2y}{dx^2} + b^2y = 0 \qquad (viii)$$

The parameters a and c have disappeared in (*viii*) and a differential coefficient of the second order has appeared. Equation (*viii*) is the differential equation of the family of curves represented by equation (*v*) Observe that the equation of the family involved two parameters and the differential equation of the family is of the second order, since two relations *e.g.* (*vi*), (*vii*) had to be obtained by differentiation twice, to eliminate the two parameters. In general the order of the equation is equal to the number of arbitrary constants eliminated by differentiation.

We shall now consider some simple differential equations arising from physical problems.

Suppose a particle is moving with a constant velocity u along the axis of x. Let x be its distance from the origin at time t. Then its velocity at time t is $\frac{dx}{dt}$. This is constant and equal to u

$$\therefore \quad \frac{dx}{dt} = u$$

This is the differeutial equation of the motion of the particle.

Consider the motion of a particle projected vertically upwards with an initial velocity u. Let x denote the distance of the particle above the point of projection O at time t. Then its velocity and acceleration at time t are

$$v = \frac{dx}{dt} \quad \text{and } a = \frac{d^2x}{dt^2}$$

Since its acceleration is only due to gravity, it is $-g$.

$$\therefore \quad \frac{d^2x}{dt^2} = -g.$$

This is the differential equation of the motion of the particle. We will see later that this equation, along with given initial conditions, completely determines the motion of the particle.

In conclusion, differential equations are formed by eliminating parameters from equations or arise in the solution of physical problems.

Illustrative Examples

Ex. 1. Form differential equations of the families of curves represented by the following equations. c is the parameter in each case.

$(i)\ xy=c^2\ (ii)\ y=cx+x^2\ (iii)\ y=cx+2\ (iv)\ e^x+ce^y=1.$

Sol. $(i)\ xy=c^2 \quad \therefore\ y + x\frac{dy}{dx} = 0$

$\therefore$ The differential equation is

$$y + x\frac{dy}{dx} = 0$$

$(ii)\ y=cx+x^2$

$$\therefore \frac{dy}{dx} = c + 2x \qquad \therefore \quad c = \frac{dy}{dx} - 2x$$

Substituting the value of c in the original equation we obtain.

$$y=\left(\frac{dy}{dx} - 2x\right)x + x^2$$

On simplification we obtain the differential equation as

$$x\frac{dy}{dx} - y = x^2$$

$(iii)\ y=cx+2$

$$\therefore \frac{dy}{dx} = c$$

Substituting the value of c in the original equation. We obtain the differential equation, as,

$$x\frac{dy}{dx} - y + 2 = 0$$

$(iv)\ e^x + ce^y = 1$

$$\therefore\ e^x + ce^y\frac{dy}{dx} = 0$$

$$\therefore\ c = -\frac{e^x}{e^y\dfrac{dy}{dx}}$$

Substituting the value of c in the original equation we get

$$e^x - \frac{e^x}{e^y\dfrac{dy}{dx}}e^y = 1$$

On simplification the differential equation is obtained as

$$(e^x - 1)\frac{dy}{dx} - e^x = 0$$

Ex. 2. Form a differential equation by eliminating the parameters c_1 and c_2 from the equation.

$$y = c_1 e^{ax} + c_2 e^{-ax}$$

Sol. $y = c_1 e^{ax} + c_2 e^{-ax}$

Differentiating successively twice we obtain,

$$\frac{dy}{dx} = c_1 a e^{ax} - c_2 a e^{-ax}$$

$$\frac{d^2y}{dx^2} = c_1 a^2 e^{ax} + c_2 a^2 e^{-ax}$$

$$\therefore \frac{d^2y}{dx^2} = a^2 (c_1 e^{ax} + c_2 e^{-ax})$$

i.e. $\frac{d^2y}{dx^2} = a^2 y$ is the differential equation.

Observe that to eliminate the two arbitrary constants c_1 and c_2 we had to differentiate twice and the result is a differential equation of the second order.

Ex. 3. Form a differential equation by eliminating the constants c_1 and c_2 from the equation.

$$y = c_1 \cos pt + c_2 \sin pt$$

Sol. $y = c_1 \cos pt + c_2 \sin pt$

Differentiating successively twice we obtain,

$$\frac{dy}{dt} = -c_1 p \sin pt + c_2 p \cos pt$$

$$\frac{d^2y}{dt^2} = -c_1 p^2 \cos pt - c_2 p^2 \sin pt$$

$$\therefore \frac{d^2y}{dt^2} = -p^2 (c_1 \cos pt + c_2 \sin pt)$$

i.e. $\frac{d^2y}{dt^2} = -p^2 y$ is the differential equation.

Ex. 4. A particle is moving along the axis of x. Its acceleration at time t is proportional to its velocity at that time, k being the constant of proportionality. Form the differential equation of the motion of the particle.

Sol. Let x be the distance of the particle from the origin at time t. Then its velocity v and acceleration a are,

$$v = \frac{dx}{dt} \text{ and } a = \frac{d^2x}{dt^2}.$$

By the condition of the motion.

$$a = kv$$

$$\therefore \frac{d^2x}{dt^2} = k \frac{dx}{dt}$$

$$i.e. \quad \frac{d^2x}{dt^2} - k\frac{dx}{dt} = 0$$

This is the differential equation of the motion of the particle.

Ex. 5. The rate of decay of the mass of a radio-active substance at any instant is α times its mass at that instant. Form the differential equation satisfied by the mass of the substance.

Sol. Let m denote the mass of the substance at time t.

Then $\frac{dm}{dt}$ = rate of change of mass m *w.r.t.* time t.

Since the mass decays *i.e.* decreases this rate of change is—ve. Also its magnitude is αm

$$\therefore \quad \frac{dm}{dt} = -\alpha m.$$

$$i.e. \quad \frac{dm}{dt} + \alpha m = 0$$

This is the differential equation satisfied by the mass of the radioactive substance.

7·4. Graphs of a sin $(bx+c)$ and a cos $(bx+c)$.

The student is familiar with the graph of $y=\sin x$ which is shown below in Fig. 7·1

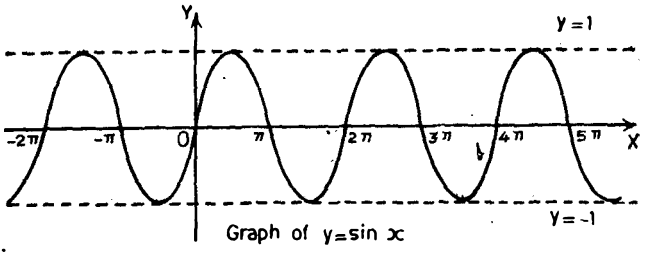

Fig. 7·1.

Now consider $y=a \sin x$.

The graph of this function is similar to the graph of $y=\sin x$, only the y coordinate of each point is a times the y coordinate of the corresponding point in the graph of $y=\sin x$.

Also $-1 \leqslant \sin x \leqslant 1 \Rightarrow -a \leqslant a \sin x \leqslant a$

$\therefore$ the graph of $y = a \sin x$ will be similar in shape to that of $y = \sin x$ and will lie between the lines

$$y = -a \text{ and } y = a$$

This graph is shown below 7.2.

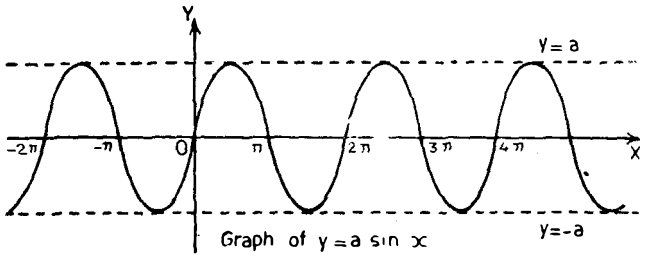

Fig. 7·2

Similarly the graph of $y = a \cos x$ will be similar to the graph of $y = \cos x$ and will lie between the lines $y = -a$ and $y = a$. It is shown below in Fig. 7.3

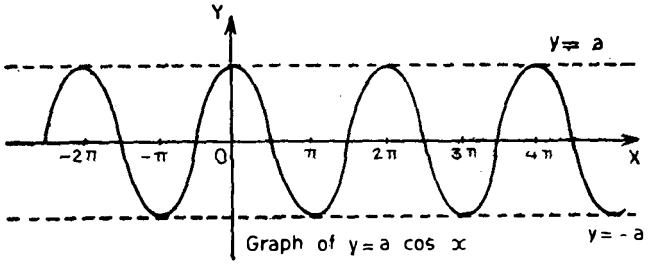

Fig. 7.3

Now consider $y = a \sin 2x$

We have $a \sin (2x + 2\pi) = a \sin 2x$

i.e. $a \sin 2(x + \pi) = a \sin 2x$

$\therefore$ $a \sin 2x$ is a periodic function with period π.

Also we may write $y = a \sin 2x$ as,

$y = a \sin \theta$ where $\theta = 2x$

$\therefore$ the graph of $y = a \sin 2x$ will be similar in shape to the graph of $y = a \sin x$ but with period π. This graph is shown below in Fig. 7.4

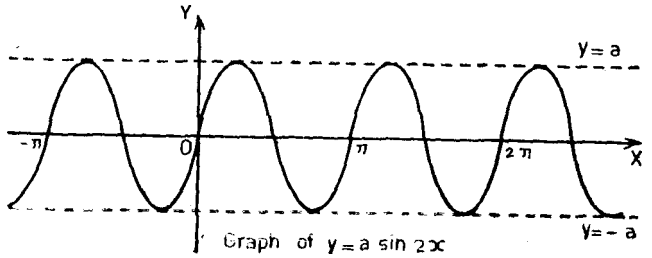

Fig 7.4

Consider now $y = a \sin bx \qquad b>0$

We have $a \sin (bx+2\pi) = a \sin bx$

$$i.e.\ a \sin b\,(x + \frac{2\pi}{b}) = a \sin bx$$

$\therefore$ $a \sin bx$ is a periodic function with period $\frac{2\pi}{b}$. As in the case of $a \sin 2x$, the graph of $y = a \sin bx$ will be similar to the graph of $y = a \sin x$ but with period $\frac{2\pi}{b}$. This graph is shown below in fig. 7·5.

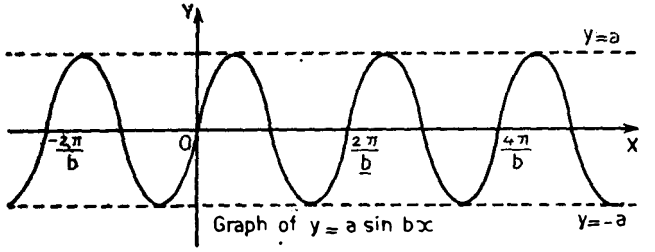

Fig. 7·5.

In a similar manner the graph of $y = a \cos bx$ will be similar to the graph of $y = a \cos x$ but with period $\frac{2\pi}{b}$. It is shown below in fig. 7·6.

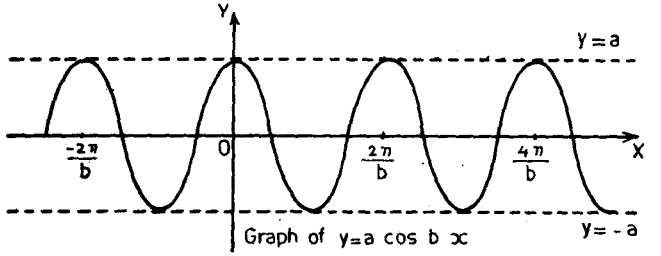

Fig. 7·6

Thus the graphs of $a \sin bx$ and $a \cos bx$ are similar to the graphs of $a \sin x$ and $a \cos x$. Only there is a change in the period. The period is changed from 2π to $\frac{2\pi}{b}$.

Finally consider, $y = a \sin (bx+c)$.

We have $y = a \sin b\left(x + \frac{c}{b}\right)$.

Shift the origin to $\left(\frac{-c}{b}, 0\right)$, which is a point on the axis of x. If (x, y) represent the coordinates of a point in the new system.

$$X = x - \left(\frac{-c}{b}\right) \qquad y = Y - 0$$

$$= x + \frac{c}{b} \qquad = y.$$

$\therefore$ the equation $y = a \sin (bx+c)$ *i.e.* $y = a \sin b\,(x+c)$ changes to $Y = a \sin bX$ in the system of coordinates.

Now the graph of $Y = a \sin bX$ in the new system is the same as that of $y = a \sin bx$ in the old system.

$\therefore$ The graph of $y = a \sin (bx + c)$ is the same as that of $y = a \sin bx$, with origin shifted to the point $\left(\frac{-c}{b}, 0\right)$. This is shown in fig. 7·7. (a and b are assumed to be positive.)

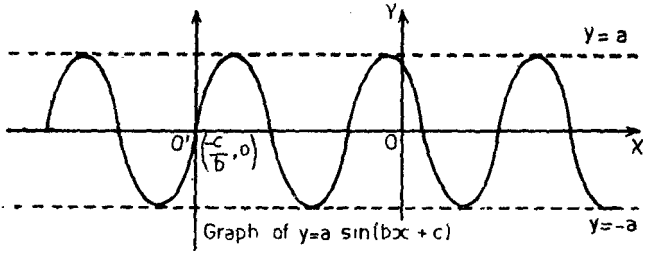

Fig. 7.7

Observe that the graph of $y = a \sin (bx + c)$ is the same curve as that of $y = a \sin bx$ translated along the axis of x through a distance $\frac{c}{b}$ to the left.

In the same way the graph $y = a \cos (bx+c)$ will be the same as the graph of $y = a \cos bx$ with the origin at $\left(-\frac{c}{b}, 0\right)$. This is shown below in fig. 7·8.

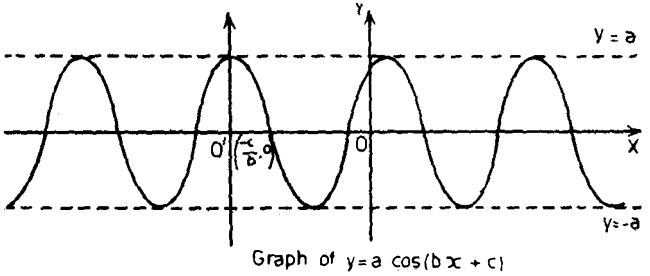

Fig. 7.8

Again the graph of $y=a\cos(bx+c)$ is the same curve as that of $y=a\cos bx$, translated along the axis of x through a distance $\frac{c}{b}$ to the left $(a, b, c \triangleright 0)$.

Fig. 7·9. show the graphs of two particular cases of the above two functions.

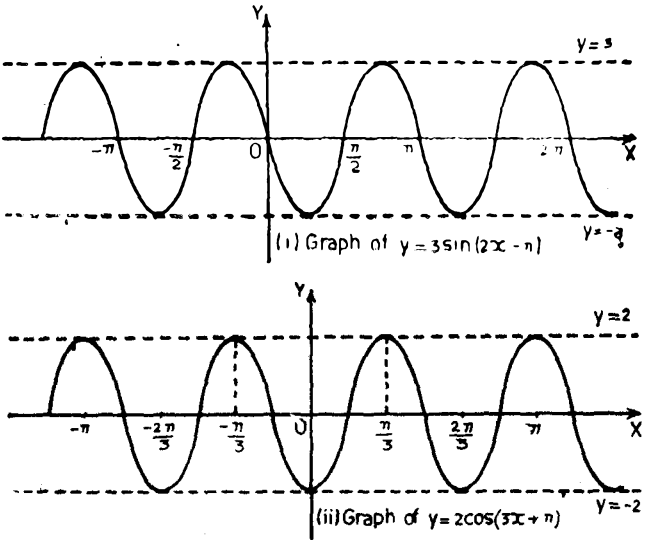

Fig. 7·9

We conclude therefore that the graphs of

$y=a\sin(bx+c)$ and $y=a\cos(bx+c)$ are of the same shape as sine and cosine curves with period $\frac{2\pi}{b}$ and shifted along the axis of x through a distance $-\frac{c}{b}$. Further whereas the graphs $y=\sin x$ and $y=\cos x$ lie between $y=-1$ and $y=+1$, the graphs of these functions lie between $y=-a$ and $y=+a$. The magnitude of a decides the size of the graphs. For different values of a and c, these equations will represent a family of curves similar in shape to the sine and cosine curves, with period $\frac{2\pi}{b}$ and shifted along the axis of x through $-\frac{c}{b}$. Observe further that both these equations represent the same family of curves since,

$$a\sin\left(bx+c+\frac{\pi}{2}\right)=a\cos(bx+c)$$

Thus we may take either of

$y = a \sin (bx + c)$ or $y = a \cos (bx + c)$ where a and c are parameters as the equation of this family of curves.

It has been seen earlier that differential equation of this family of curves is,

$$\frac{d^2y}{dx^2} + b^2y = 0$$

EXERCISE : 7·1

1. State the order and the degree of the following differential equations :

(*i*) $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + 4y = 0$;

(*ii*) $\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} = 0$;

(*iii*) $(3x + y)\frac{dy}{dx} + y = 0$;

(*iv*) $\frac{d^2y}{dx^2} - a^2y = 0$;

(*v*) $x^2\left(\frac{dy}{dx}\right)^2 + y^2 - 1 = 0$.

2. Form differential equations by eliminating the constant c from the following equations :

(*i*) $y = cx^2$; (*ii*) $x^2 + cy^2 = a^2$; (*iii*) $cy + \log x = 0$;
(*iv*) $y = a(x - c)^2$; (*v*) $y = c\mathrm{T}anx$; (*vi*) $y = ce^{-x}$
(*vii*) $e^x + ce^y = 1$; (*viii*) $y = cx - c^2$; (*ix*) $(x + y)^2 + 2cx = 0$;
(*x*) $(x - c)^2 + y^2 = 4$; (*xi*) $a + bx + cx^2 = y$

3. Write down the equation of the family of lines having y-intercept c. Find the differential equation of this family.

4. Show that $x^2 + y^2 - 2c(x + y) = 0$ represents a family of circles passing through the origin and with centre on the line $y = x$. Eliminate c and form the differential equation of the family.

5. Write down the differential equation of the motion of a particle moving along the axis of x in such a way that its velocity at time t is proportional to its distance from the origin at that time.

6. Form the differential equations from the following equations by eliminating the constants c_1 and c_2.

(*i*) $y = c_1 e^{2x} + c_2 e^{-2x}$, (*ii*) $y = c_1 e^x + c_2$
(*iii*) $y = c_1 x^2 + c_2 x$, (*iv*) $y = c_1 x + c_2$
(*v*) $y = e^x (c_1 + c_2 x)$; (*vi*) $y = c_1 \mathrm{Sin}\,(px + c_2)$.

ANSWERS

1. (*i*) 3,1 ; (*ii*) 2,2 ; (*iii*) 1,1 : (*iv*) 2,1 ; (*v*) 1,2.

2. (*i*) $x\frac{dy}{dx} - 2y = 0$; (*ii*) $(x^2 - a^2)\frac{dy}{dx} - xy = 0$;

(*iii*) $x \log x \frac{dy}{dx} - y = 0$ (*iv*) $\left(\frac{dy}{dx}\right)^2 = 4ay$;

(*v*) $\text{Sin } 2x \frac{dy}{dx} - 2y = 0$; (*vi*) $\frac{dy}{dx} + y = 0$;

(*vii*) $(e^x - 1) \frac{dy}{dx} - e^x = 0$; (*viii*) $y = x \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$

(*ix*) $(x+y) + 2x \frac{dy}{dx} = 0$; (*x*) $y^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right] = 4$;

(*xi*) $2y - x \frac{dy}{dx} = 2a + bx$

7.5. Solution of a differential equation.

A solution of a differential equation is a relation not involving differential coefficients such that, this relation along with the differential coefficients obtained from it satisfy the differential equation.

For example consider the differential equation,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2y$$

The relation $y = e^x$ is a solution of this equation because we have,

$$\frac{dy}{d\alpha} = e^x \; ; \quad \frac{d^2y}{dx^2} = e^x.$$

$$= y \qquad = y$$

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} = 2y$$

Similarly $y = e^{-2x}$ is also a solution because,

$$\frac{dy}{dx} = -2e^{-2x} \qquad \frac{d^2y}{dx^2} = 4e^{-2x}$$

$$= -2y \qquad = 4y$$

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} = 2y$$

The student should verify that $y = c_1e^x + c_2e^{-2x}$ where c_1 and c_1 are any numbers is also a solution of the differential equation. c_1 and c_2 are called arbitrary constants.

7.6. Types of solutions.

There are two types of solutions of a differential equation :

(*i*) *General solution or primitive.* A general solution is a solution which contains arbitrary constants (such as c_1 and c_2 in the above example) whose number is equal to the order of the equation.

(*ii*) *Particular solution.* A particular solution of a differential equation is a solution of the differential equation obtained from its general solution by giving particular values to the arbitrary contants in the general solution.

Thus $y=c_1e^x+c_2e^{-2x}$ is the general solution of $\frac{d^2y}{dx^2}+\frac{dy}{dx}=2y$. Putting $c_1=1$ & $c_2=0$ we obtain $y=e^x$ which is a particular solution. Similarly putting $c_1=0$ and $c_2=1$ we obtain the other particular solution *e.g.* $y=e^{-2x}$.

7.7. Differential equation of the first order and the first degree.

From the examples considered so far we may writn the general differential equation of the first order and the first degree as

$$P(x,y)+Q(x,y)\frac{dy}{dx}=0$$

In practice it is convenient to write this equation in the form

$$P(x,y)\,dx+Q(x,y)\,dy=0$$

or $P\,dx+Q\,dy=0$

where P and Q are functions of x and y.

Thus (*i*) $(2x+y)+(y-x)\frac{dy}{dx}=0$

is written as

$$(2x+y)\,dx+(y-x)\,dy=0$$

(*ii*) $(x+y)\frac{dy}{dx}+(x-y)=0$

is written as

$$(x-y)\,dx+(x+y)\,dy=0$$

7.8. Method of solving the differential equation when the variables are separable.

Consider the differential equation,

$$y^2dx+x^2dy=0$$

Dividing by x^2y^2 we obtain

$$\frac{dx}{x^2}+\frac{dy}{y^2}=0$$

We can now integrate the equation as,

$$\int\frac{dx}{x^2}+\int\frac{dy}{y^2}=c$$

i.e. $$-\frac{1}{x}-\frac{1}{y}=c$$

i.e. $$x+y+cxy=0$$

This relation contains one arbitrary constant *e.g. e.* Since the differential equation is of order one,

$x+y+cxy=0$ is the general solution of $y^2dx+x^2dy=0$. Note that the general solution of a differential equation of the first order will contain one arbitrary constant.

In this example we were able to separate the two variables x and y and direct integration *w.r.t.* these was possible. Such equations are called *equations with separable variables.* We shall consider only this type of differential equations.

The general form of the differential equation of first order and the first degree with separable variables is,

$P_1(x).\ P_2(y)\ dx + Q_1(x).\ Q_2(y)\ dy = 0$

For values of x and y for which $Q_1(x) \neq 0$ and $P_2(y) \neq 0$ we can write the equation as

$$\frac{P_1(x)}{Q_1(x)}\ dx + \frac{Q_2(y)}{P_2(y)}\ dy = 0$$

The variables are now separated and we can integrate the equation as

$$\int \frac{P_1(x)}{Q_1(x)} dx + \int \frac{Q_2(y)}{P_2(y)}\ dy = c$$

where c is an arbitrary constant. This relation contains one arbitrary constant and hence is the general solution.

Illustrative Examples

Ex. 1. Solve the following differential equations:

(*i*) $\sqrt{1-y^2}\ dx + \sqrt{1-x^2}\ dy = 0$

(*ii*) $(1+y)\ xdx + y\ (1-x)\ dy = 0$

(*iii*) $\sin x \cos y dx + \cos x \sin y dy = 0$

(*iv*) $(1+y^2)\ e^x\ dx + (e^x + 1)\ dy = 0$

Sol. (*i*) $\sqrt{1-y^2}\ dx + \sqrt{1-x^2}\ dy = 0$

Dividing by $\sqrt{1-y^2}\ \sqrt{1-x^2}$ we get,

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} = 0$$

$\therefore$ Integrating

$$\int \frac{dx}{\sqrt{1-x^2}} + \int \frac{dy}{\sqrt{1-y^2}} = c$$

$\therefore\ \sin^{-1}x + \sin^{-1}y = c$

This is the general solution.

(*ii*) $(1+y)\ xdx + y\ (1-x)\ dy = 0$

Dividing by $(1+y)\ (1-x)$ we get,

$$\frac{xdx}{1-x} + \frac{ydy}{1+y} = 0$$

$\therefore$ Integrating

$$\int \frac{xdx}{1-x} + \int \frac{ydy}{1+y} = c$$

$$\therefore \int \frac{-(1-x)+1}{1-x}\,dx + \int \frac{(1+y)-1}{1+y}\,dy = c.$$

$$\therefore \int\left[-1+\frac{1}{1-x}\right]dx + \int\left[1-\frac{1}{1+y}\right]dy = c$$

$$\therefore -\int 1\,dx + \int\frac{dx}{1-x} + \int 1\,dy - \int\frac{dy}{1+y} = c$$

$$\therefore -x - \log(1-x) + y - \log(1+y) = c.$$

$\therefore y-x-\log(1-x)(1+y)=c$ is the general solution.

(*iii*) $\sin x \cos y\,dx + \cos x \sin y dy = 0$

Dividing by $\cos x \cos y$ we get,

$$\frac{\sin x}{\cos x}\,dx + \frac{\sin y}{\cos y}\,dy = 0$$

i.e. $\text{Tan}\,x\,dx + \text{Tan}\,y\,dy = 0$

$$\int \text{Tan}\,x\,dx + \int \text{Tan}\,y dy = c$$

$$\therefore \log \sec x + \log \sec y = c$$

$$\therefore \log \sec x \sec y = c$$

$$\therefore \sec x \sec y = e^c$$

Since c is arbitrary e^c is also an arbitrary constant.

Writing $c_1 = e^c$ the solution is

$$\sec x \sec y = c_1$$

$$\text{or} \quad \cos x \cos y = G \quad \text{where} \quad G_1 = \frac{1}{c_1}$$

(*iv*) $(1+y)^2 e^x\,dx + (e^x+1)\,dy = 0$

Dividing by $(1+y^2)(e^x+1)$ we get,

$$\frac{e^x}{e^x+1}\,dx + \frac{dy}{1+y^2} = 0$$

$$\therefore \int\frac{e^x}{e^x+1}\,dx + \int\frac{dy}{1+y^2} = c$$

$$\therefore \log(e^x+1) + \text{Tan}^{-1}y = c \qquad (1)$$

$$\therefore \log(e^x+1) = c - \text{Tan}^{-1}y$$

$$\therefore e^x + 1 = e^{c-\text{Tan}^{-1}y}$$

$$= e^c - e^{-\text{Tan}^{-1}y}$$

$$\therefore e^x + 1 = c_1 e^{-\text{Tan}^{-1}y} \quad \text{where } c_1 = e^c$$

$$\text{or} \quad e^x = ce_1 - \text{Tan}^{-1}y - 1 \qquad \ldots (2)$$

(1) and (2) are both solutions of the differential equation. Usually the general solution of a differential equation can be expressed in more than one equivalent forms.

Ex. 2. Solve $2ax\,dx - dy = 0$

What is the family of curves represented by the general solution of this equation ? Find the equation of the member of this family passing through (2, 0.).

Sol. $2a\,x\,dx - dy = 0$

$\therefore \int 2ax\,dx - \int dy = c$

$\therefore \quad ax^2 - y = c$

or $\quad y = ax^2 - c$

This equation represents a family of parabolas. The value of c for which this will pass through (2, 0) will be determined by

$$0 = a\,(2)^2 - c$$

$\therefore \quad c = 4a$

$\therefore$ the member of this family which passes through (2, 0) is,

$$y = ax^2 - 4a$$

Ex. 3. The acceleration of a particle moving along the axis of x is $6t + 4$ at time t. If the particle starts from the origin with velocity 5, find x, its distance from the origin at time t, as a function of t.

Sol. Let $x =$ distance of the particle from the origin at time t.

$\therefore$ if v is the velocity at time t and a the acceleration at that time.

$$v = \frac{dx}{dt} \text{ and } \frac{dv}{dt} = a$$

Now the acceleration at time t is $6t+4$.

$\therefore \quad \frac{dv}{dt} = 6t+4$

$\therefore \quad dv = (6t+4)dt$

$\therefore \quad \int dv = \int (6t+4)\,dt + c$

$\therefore \quad v = 3t^2+4t+c$

The particle starts with velocity 5

$\therefore \quad v = 5$ when $t = 0$

$\therefore \quad 5 = 3(0)^2 + 4(0) + c$ *i.e.* $c=5$.

$\therefore \quad v = 3t^2+4t+5$

$\therefore$ *i.e.* $\quad \frac{dx}{dt} = 3t^2+4t+5$

$\therefore \quad dx = (3t^2+4t+5)dt$

$\therefore \quad \int dx = \int (3t^2+4t+5)dt + c$

$\therefore \quad x = t^3+2t^2+5t+c$

The particle starts from the origin

$\therefore \quad x = 0$ when $t = 0$

$\therefore \quad 0 = (0)^3+2(0)^2+5(0)+c$, *i.e.* $c=0$

$\therefore \quad x = t^3+2t^2+5t$

Ex. 4. Obtain the solution of the differential equation $\frac{dy}{dx} - \alpha y = 0$, which satisfies $y = a$ when $x=0$

Sol. $\frac{dy}{dx} - \alpha y = 0$

$\therefore \quad dy - \alpha y\, dx = 0$

$\therefore \quad \frac{dy}{y} - \alpha dx = 0$

Integrating

$$\log y - \alpha x = c_1$$

$\therefore \quad \log y = \alpha x + c_1$

$\therefore \quad y = e^{\alpha x + c_1}$

$$= e^{c_1} e^{\alpha x}$$

$\therefore \quad y = ce^{\alpha x}$ where $c = e^{c_1}$.

This is the general solution.

Now $y=a$ when $x=0$,

$\therefore \quad a = ce^{\alpha(0)}$

$$= c$$

$\therefore \quad y = ae^{\alpha x}$ is the required solution.

Ex. 5. Solve the differential equation,

$$x\, dx + y dy = 0$$

and interpret the general solution.

Sol. $x dx + y\, dy = 0$

Here the variables are separated. We get on integration,

$$\int x\, dx + \int y\, dy = c,$$

i.e. $\frac{1}{2}x^2 + \frac{1}{2}y^2 = c_1$

i.e. $x^2 + y^2 = 2c_1$

$\therefore \quad x^2 + y^2 = c$ is the general solution.

For any given value of c this equation represents a circle with origin as the centre and radius $\sqrt{c}$.

$\therefore \quad x^2 + y^2 = c$ where c is a parameter which represents a family of circles having origin as centre.

7·9. Equations reducible to the variables separable form.

Some times an equation in which the variables are not separable can be reduced to an equation with separable variables by a suitable substitution. The solution of this second equation then leads to the solution of the original equation on substituting the value of the new variable in terms of x and y. This is illustrated in the examples below.

Ex. 1. Solve $(x-y)\left(1-\frac{dy}{dx}\right)=e^x$ by substituting $u=x-y$.

Sol. $(x-y)\left(1-\frac{dy}{dx}\right)=e^x$

$u=x-y$

$\therefore \quad \frac{du}{dx}=1-\frac{dy}{dx}$

$\therefore$ The equation becomes,

$$u\frac{du}{dx}=e^x$$

i.e. $\quad udu=e^xdx$

Integrating we get,

$$\tfrac{1}{2}u^2=e^x+c$$

Substituting back the value of u in terms of x and y, we get,

$\frac{1}{2}(x-y)^2=e^x+c$

This is the required solution.

Ex. 2. Solve $\left(x\frac{dy}{dx}-y\right)\sin\frac{y}{x}=x^2\cos x$ by, substituting $u=\frac{y}{x}$.

Sol. $\left(x\frac{dy}{dx}-y\right)\sin\frac{y}{x}=x^2\cos x$

$\therefore \quad \frac{x\frac{dy}{dx}-y}{x^2}\sin\frac{y}{x}=\cos x$

Now $\quad u=\frac{y}{x}$

$\therefore \quad \frac{du}{dx}=\frac{x\frac{dy}{dx}-y}{x^2}$

$\therefore$ The equation reduces to,

$$\frac{du}{dx}\sin u=\cos x$$

$\therefore \quad \sin udu=\cos xdx$

Integrating we get

$$-\cos u = \sin x + c$$

i.e. $\sin x + \cos u + c = 0$

Substituting back the value of u in terms of x and y we get,

$$\sin x + \cos \frac{y}{x} + c = 0$$

This is the general solution of the given equation.

Ex. 3. Solve the following differential equations by substituting $y = ux$,

(*i*) $x \frac{dy}{dx} = x + 2y$; (*ii*) $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

Sol. (*i*) $x \frac{dy}{dx} = x + 2y$

$$y = ux$$

$$\therefore \frac{du}{dx} = u + x \frac{du}{dx}$$

$\therefore$ The given equation reduces to,

$$x\left(u + x\frac{du}{dx}\right) = x + 2ux$$

$$\therefore u + x\frac{du}{dx} = 1 + 2u$$

i.e. $x\frac{du}{dx} = 1 + u$

$$\therefore \frac{du}{1+u} = \frac{dx}{x}$$

Integrating we get,

$$\log (1 + u) = \log x + c_1$$

i.e. $\log (1 + u) - \log x = c_1$

$$\therefore \log \frac{1+u}{x} = c_1$$

$$\therefore \frac{1+u}{x} = e^{c_1}$$

i.e. $\frac{1+u}{x} = c$ where $e^{c_1} = e$

Substituting back the value of u in terms of x and y we get,

$$\frac{1 + \frac{y}{x}}{x} = c$$

i.e. $x + y = c\,x^2$

This is the solution of the given equation.

(*ii*) $x^2 \dfrac{dy}{dx} = x^2+xy+y^2$

$$y = ux$$

$$\therefore \frac{dy}{dx} = u + x\frac{du}{dx}.$$

Substituting in the given equation,

$$x^2\left(u + x\frac{du}{dx}\right) = x^2+xux+u^2x^2$$

$$i.e.\ x^2\left(u+x\frac{du}{dx}\right) = x^2(1+u+u^2)$$

$$\therefore \quad u + x\frac{du}{dx} = 1+u+u^2$$

$$\therefore \quad x\frac{du}{dx} = 1 + u^2$$

$$\therefore \quad \frac{du}{1+u^2} = \frac{dx}{x}$$

Integrating we get,

$$\text{Tan}^{-1}u = \log x + c$$

Putting back the value of u in terms of x and y we get, the solution of the original equation as

$$\text{Tan}^{-1}\frac{y}{x} = \log x + c.$$

Ex. 4. Solve $\cos(x+y)\dfrac{dy}{dx} = 1$ by substituting $x+y=v$

Sol. $\cos(x+y)\dfrac{dy}{dx} = 1$

$$x + y = v$$

$$\therefore \quad 1+\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$\therefore$ the given equation reduces to

$$\cos v\left(\frac{dv}{dx} - 1\right) = 1$$

$$i.e.\ \cos v\ \frac{dv}{dx} = 1 + \cos v$$

$$\therefore \quad \frac{\cos v\ dv}{1+\cos v} = dx$$

Integrating we get

$$\int \frac{\cos v\ dv}{1+\cos v} = \int dx + c$$

$$\therefore \int \frac{(1+\cos v)-1}{1+\cos v}\ du = \int dx + c$$

$$\therefore \int\left[1 - \frac{1}{1+\cos v}\right] dv = x + c$$

$$\therefore \int dv - \int \frac{dv}{2\cos^2 \frac{u}{2}} = x + c$$

$$\therefore \quad v - \int \operatorname{Sec}^2 \frac{v}{2} \cdot \frac{dv}{2} = x + c$$

$$\therefore \quad v - \tan \frac{v}{2} = x + c$$

Substituting back the value of v in terms of x and y.

$$x + y - \tan \frac{x+y}{2} = x + c$$

$\therefore$ $y - \tan \frac{x+y}{2} = c$ is the solution of the original equation.

Ex. 5. Solve $\frac{dy}{dx} = \cos(x+y)$

Sol. $\frac{dy}{dx} = \cos(x+y)$

Write $x + y = v$

$$\therefore \quad 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$\therefore$ the equation reduces to,

$$\frac{dv}{dx} - 1 = \cos v$$

i. e. $\frac{dv}{dx} = 1 + \cos v$

$$\therefore \quad \frac{dv}{1+\cos v} = dx$$

i. e. $\frac{dv}{2\cos^2 \frac{v}{2}} = dx$

$$\therefore \quad \tfrac{1}{2} \sec^2 \frac{v}{2}\, dv = dx$$

Integrating we get,

$$\tan \frac{v}{2} = x + c$$

Substituting the value of v, we get,

$$\tan \frac{x+y}{2} = x + c$$

This is the solution of the original equation.

EXERCISE 7.2

1. Solve the following differential equations.

(*i*) $(x-1)\,dx + (y-2)\,dy=0$; (*ii*) $y^2\,dx-(x-1)^2\,dy=0$;

(*iii*) $dx + y^2\,xdy=0$; (*iv*) $\sin x\,dx + \sec^2 y \cos x\,dy=0$;

(*v*) $xe^{-y}dx + ye^{x^2}dy=0$ (*vi*) $\sec y\,dx - xy \log x\,dy = 0$;

(*vii*) $xydx + \sqrt{1+x^2}\,dy=0$ (*viii*) $\frac{y}{x}\,\frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}=0$;

(*ix*) $\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy=0$

(*x*) $(1+y^2)\sin^{-1} x\,dx + \sqrt{1-x^2}\,dy=0$ (*xi*) $\frac{dy}{dx} = e^{x+y}$

(*xii*) $y-x\,\frac{dy}{dx} = y^2 + \frac{dy}{dx}$; (*xiii*) $\frac{dy}{dx} = \frac{xy}{(1-x)(1+y)}$;

(*xiv*) $(x^2-yx^2)\,\frac{dy}{dx} + y^2+x^2y^2=0$

(*xv*) $\sec^2 x \sin y dx + (\tan x+1)\cos y\,dy = 0$

(*xvi*) $x\,(1+y^2)\,dx + y\,(1+x^2)\,dy=0$;

(*xvii*) $(x-y^2x)\,dx-(y-x^2y)\,dy=0$

(*xviii*) $(1-x^2)\,(1-y)\,dx-xy\,(1+y)\,dy=0$

(*xix*) $\frac{dy}{dx}\tan y = \sin(x+y) - \sin(x-y)$

2. Solve $\frac{dy}{dx} + xy=xy^2$ where $y=4$ when $x=1$

3. Find the solution of $(x+1)\,(y-1)\,dx + (x-1)\,(y+1)\,dy=0$. such that, $y=2$ when, $x=2$

4. Solve $\frac{dy}{dx} = 2y\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ where $y=1$ when $x=0$

5. Solve $y\,(\log x+1)\,\frac{dy}{dx} - x\log x = 0$ and find a solution such that $y=e^2$ when $x=e$.

6. Solve $(xy+x+2y+2)\,dz-xy\,dy=0$ where $y=0$ when $x=1$

7. Solve $(1-x)\,dy-(1+y)\,dx=0$ and find the particular solution such that $y=4$ when $x=2$.

8. Find the particular solution of $\frac{dy}{dx} = -\frac{y+2}{x+2}$ such that $y=2$ when $x=-1$.

9. Solve $\frac{dy}{dx} = e^y \cos x$ and find the particular solution such that $y=0$ when $x=0$.

10. Find the particular solution of.

$$y\sqrt{1-x^2}\,dy + x\sqrt{1-y^2}\,dy=0$$

such that $y=\frac{4}{5}$ when $x=\frac{3}{5}$.

11. Solve :

(*i*) $1+\frac{dy}{dx} = \operatorname{cosec}(x+y)$ by substituting $v=x+y$.

(*ii*) $(x+y)\,\frac{dy}{dx} + y=0$ by putting $x+y=v$.

(*iii*) $x\,dx+y\,dy=x^2+y^2$ by writing $u=x^2+y^2$.

(*iv*) $\left(y+x\dfrac{dy}{dx}\right)\sin xy=\cos x$ by substituting $xy=v$.

(*v*) $\left(x\dfrac{dy}{dx}-y\right)e^{y/x}=x^2\cos x$ by substituting $v=\dfrac{y}{x}$

(*vi*) $(x-y)^2\dfrac{dy}{dx}=a^2$ by putting $x-y=v$.

(*vii*) $\cos^2(x-2y)=1-2\dfrac{dy}{dx}$ by substituting $x-2y=u$

(*viii*) $(4x+y)^2\dfrac{dy}{dx}=1$ by writing $4x+y=v$.

(*ix*) $\dfrac{dy}{dx}=\dfrac{x+y+1}{2x+2y-1}$ by putting $x+y=v$.

(*x*) $(x-y)\left(1-\dfrac{dy}{dx}\right)=e^x$ by putting $v=x-y$.

12. Solve the following differential equations by substituting $y=vx$.

(*i*) $x\dfrac{dy}{dx}=2x+3y$; (*ii*) $x\dfrac{dy}{dx}=3y-x$;

(*iii*) $x\dfrac{dy}{dx}=2y-3x$; (*iv*) $xy\dfrac{dy}{dx}=x^2+y^2$;

(*v*) $x^2\dfrac{dy}{dx}=xy-y^2$; (*vi*) $xy\dfrac{dy}{dx}=2x^2+xy+y^2$.

ANSWERS

1. (*i*) $(x-1)^2+(y-2)^2=c$; (*ii*) $x-y-1=cy\,(x-1)$

(*iii*) $x=ce^{-\frac{1}{3}x^3}$; (*iv*) $4\tan y-\cos 2x=c$

(*v*) $2e^y(y-1)=c+e^{-x^2}$; (*vi*) $\log(\log x)-\sin y=c$

(*vii*) $y=c\,e^{-\sqrt{1+x^2}}$ (*viii*) $\sqrt{1-x^2}+\sqrt{1-y^2}=c$;

(*ix*) $\sin^{-1}x+\cos^{-1}y=c$; (*x*) $(\sin^{-1}x)^2+2\tan^{-1}y=c$:

(*xi*) $e^{x+y}+c\,e^y=1$; (*xii*) $(x+1)=\dfrac{cy}{1-y}$;

(*xiii*) $y+\log y=c-x-\log(1-x)$ (*xiv*) $x-\dfrac{1}{x}-\dfrac{1}{y}-\log y=c$;

(*xv*) $(1+\tan x)\sin y=c$ (*xvi*) $(1+x^2)(1+y^2)=e$;

(*xvii*) $(1-y^2)=c(1-x^2)$;

(*xviii*) $\log\dfrac{x}{1-y}=c+\dfrac{1}{2}(x^2+y^2+4y)$;

(*xix*) $\log\tan\left(\dfrac{y}{2}+\dfrac{\pi}{4}\right)=2\sin x+c$

2. $\log\dfrac{y-1}{y}=\dfrac{1}{2}x^2+\log 2-\dfrac{1}{2}$

3. $(x+y)+2\log(x-1)(y-1)=4$; 4. $y=\dfrac{1}{2}(e^{2x}+e^{-2x})$;

5. $y=ex\log x$; 6. $y+1=x^2e^{1+y-x}$

7. $(1-x)(1+y)=c$; $(1-x)(1+y)+5=0$

$(x+2)(y+2)=4$; 9. $\sin x+e^{-y}=1$

10. $5\sqrt{1-y^2}+5\sqrt{1-x^2}=7$;

11. (*i*) $x+\cos(x+y)=c$; (*ii*) $y^2+2xy=c$

(*iii*) $x^2+y^2=ce^{2x}$ (*iv*) $\sin x+\cos xy=c$

(*v*) $e^{y/x}=\sin x+c$; (*vi*) $y=\frac{a}{2}\log\frac{x-y-a}{x-y+a}+c$;

(*vii*) $x=\tan(x-2y)+c$;

(*viii*) $x\equiv\frac{1}{4}[(4x+y)-\frac{1}{2}\operatorname{Tan}^{-1}(4x+y)]+e$

(*ix*) $x=\frac{2}{3}(x+y)-\frac{1}{3}\log(x+y)+e$

(*x*) $(x-y)^2=2e^x+c$

12. (*i*) $x+y=cx^3$; (*ii*) $y-x=cx^2$; (*iii*) $y-3x=cx^2$

(*iv*) $x=ce^{y/x}$; (*v*) $x=ce^{y/x}$; (*vi*) $y+2x=ce^{x/y}$

7·10. Differential equations of the second order

Any general discussion of the differential equations of the second order is beyond the syllabus of the 12th standard. We shall limit ourselves to the following remarks :

(*i*) The equation has a general solution which contains two arbitrary constants.

(*ii*) All other solutions are particular solutions obtained by assigning particular values to one or more of the arbitrary constants in the general solution.

We shall obtain two differential equations of the second order and solve them.

7·11. Motion under gravity.

Suppose a particle is projected vertically upwards from a point O (not necessarily at the ground level). Let OX be the vertical through O. Then the particle moves along OX. Let x be the distance of the particle from O at time t. Then its velocity v and acceleration a at time t are,

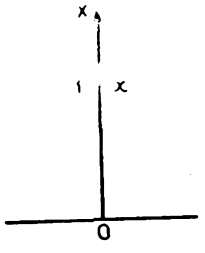

Fig. 7.10.

$$v=\frac{dx}{dt} \text{ and } a=\frac{d^2x}{dt^2}$$

Since the particle is moving freely under gravity, its acceleration is g downwards. Since x is being measured upwards the acceleration is $-g$

$$\therefore \quad a=-g \text{ i.e. } \frac{d^2x}{dt^2}=-g$$

$$\therefore \quad \frac{d^2x}{dt^2}=-g \qquad (1)$$

is the differential equation of the motion of the particle,

Equation (1) can be integrated directly w. r. t. t. since $\frac{d^2x}{dt^2}$ is the derivative of $\frac{dx}{dt}$ w.r.t.t.

$$\therefore \quad \int \frac{d^2x}{dt^2}\, dt = -\int g\, dt + c_1$$

$$\therefore \quad \frac{dx}{dt} = -gt + c_1, \; c_1 \text{ is the constant of integration } i.e.$$

$$u = c_1 - gt.$$

This determines the velocity at time t. The arbitrary constant c_1 can be determined if the velocity at any particular value of t is known.

$$\text{Now} \qquad \frac{dx}{dt} = c_1 - gt \qquad (2)$$

$$\therefore \quad dx = (c_1 - gt)\, dt$$

This is a differential equation of the first order and the first degree in which the variables are separated.

$$\therefore \quad \int dx = \int (c_1 - gt)\, dt + c_2$$

$$\therefore \quad x = c_1 t - \tfrac{1}{2} gt^2 + c_2 \qquad (3)$$

Equation (3) is a solution of the differential equation (i). It contains two arbitrary constants c_1 and c_2 and hence is the general solution.

Equations (2) and (3) give v and x at time t. They determine the motion completely. Given the initial conditions or the values of x and v for any particular value of t the values of c_1 and c_2 can be obtained. The resulting relation will be a particular case of (3) and will be a particular solution of (1).

Thus if the particle starts from the origin with velocity u, we have,

when $t = 0$, $v = u$ and $x = 0$

$\therefore$ (2) gives $u = c_1 - g(0)$ *i.e.* $c_1 = u$

$\therefore$ (3) now is

$$x = ut - \tfrac{1}{2} gt^2 + c_2$$

$\therefore$ $0 = u\,(0) - \frac{1}{2}\, g\,(0)^2 + c_2$ *i.e.* $c_2 = 0$

$\therefore$ (2) and (3) reduce to,

$$v = u - gt \text{ and } x = ut - \tfrac{1}{2} gt^2$$

The last relation is a particular solution of (1).

Note If the particle is projected downwards from the point O, then the differential equation of its motion is,

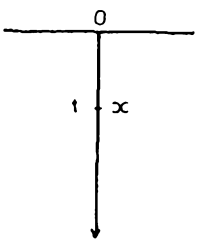

Fig. 7.11

$$\frac{d^2x}{dt^2} = g$$

Proceeding as above we obtain,

$$u = c_1 + gt \text{ and } x = c_1t + \tfrac{1}{2}gt^2 + c_2$$

In particular if it is released from rest at O, we have at $t = 0$, $x = 0$ and $v = 0$

$\therefore$ $c_1 = 0$ and $c_2 = 0$

$\therefore$ at time t, $v = gt$ and $x = \frac{1}{2}gt^2$.

Illustrative Examples

Ex. 1. A body projected vertically upwards from A, the top of a tower, reaches the grounds in t_1 second. If it is projected vertically downwards from A with the same velocity it reaches the ground in t_2 seconds. Prove that if it falls freely from A it will reach the ground in $\sqrt{t_1t_2}$ seconds.

Sol. Let h be the height of the tower and u the velocity of projection. In the first case the stone is projected vertically upwards with a velocity u. Its distance x above A at time t then is given by,

$$x = ut - \tfrac{1}{2}gt^2$$

It reaches the ground, a distance h below A in time t_1

$\therefore$ at $t = t_1$, $x = -h$

$\therefore$ (ground is below 0)

$\therefore$ $-h = ut_1 - \frac{1}{2}gt_1^2$

i.e. $h = -ut_1 + \frac{1}{2}gt_1^2$ ...I

In the second case it is projected downwards from A.

$\therefore$ if x is the distance of the particle vertically below A at time t,

$$x = ut + \tfrac{1}{2}gt^2$$

At $t = t_2$ it reaches the ground *i.e.* $x = h$ at $t = t_2$

$\therefore$ $h = ut_2 + \frac{1}{2}gt_2^2$ ...2

If the stone is released from rest at A and falls freely to the ground in 't' seconds

$\therefore$ $h = 0 \cdot t + \frac{1}{2}gt^2$

i.e. $h = \frac{1}{2}gt^2$ ...3

multiplying (1) by t_2 and (2) by t_1 and adding we get,

$$h(t_2 + t_1) = \tfrac{1}{2}g\,t_1\,t_2\,(t_1 + t_2)$$

$\therefore$ $h = \frac{1}{2}g\,t_1\,t_2$

From (3) and (4) we get,

$$\tfrac{1}{2} gt^2 = \tfrac{1}{2} g\, t_1\, t_2$$

$$\therefore \quad t = \sqrt{t_1 \cdot t_2}$$

Ex. 2. From a baloon which is ascending with a velocity of 981 cm/sec, a stone is let fall and reaches the ground in 16 seconds. How high was the baloon when the stone was dropped.

Sol. Let h be the height of the baloon above the ground when the stone was dropped. The velocity at that instant was 981 cm/sec. upwards *i.e.* —981 cm/sec. downwards. We may consider the stone to have fallen a distance h freely under gravity in 16 seconds starting with a velocity—981 cm/sec.

If x is its distance below the point of release at time t (t being measured from the instant of release).

$$x = (-981)t + \tfrac{1}{2} gt^2$$

$$\therefore \quad h = (-981)\ 16 + \tfrac{1}{2}\ (981)\ (16)^2$$

$$= 16 \times 981\ (-1 + 8)$$

$$= 109872 \text{ cms}$$

$$\therefore \quad h = 1098{\cdot}72 \text{ metres.}$$

Ex. 3. A particle is projected vertically upwards with a velocity 9.8 m/sec. Find the maximum height attained by it.

Sol. The differential equation of the motion of the particle is

$$\frac{d^2x}{dt^2} = -g \qquad \ldots 1$$

where x is the distance of the particle above the point of projection at time t.

$$\therefore \quad \int \frac{d^2x}{dt^2}\, dt = -\int g dt + c_1$$

$$\therefore \quad \frac{dx}{dt} = -gt + c_1$$

i.e. $v = -gt + c_1$ where v is the velocity at time t.

It started with velocity 9·8 m/sec. *i.e.* $v = 9{\cdot}8$ at $t = 0$.

$$\therefore \quad 9{\cdot}8 = -g(0) + c_1 \qquad i.e\ c_1 = 9{\cdot}8$$

$$\therefore \quad \frac{dx}{dt} = 9{\cdot}8 - gt \ i.e.\ v = 9{\cdot}8 - gt.$$

$$\therefore \quad dx = (9{\cdot}8 - gt)\, dt$$

$$\therefore \quad \int dx = \int (9{\cdot}8 - gt)\, dt + c_2$$

$$\therefore \quad x = (9{\cdot}8)t - \tfrac{1}{2} gt^2 + c_2$$

Since x is measured from the point of projection $x = 0$ when $t = 0$.

$\therefore \quad c_2 = 0$

$\therefore \quad x = (9.8)\, t - \frac{1}{2} gt^2$

When the particle attains the highest point $v = 0$.

$\therefore \quad 0 = 9.8 - gt$

$\therefore \quad t = \frac{9.8}{g} = 1 \qquad$ taking $g = 9.8$ ms/sec^2

At this instant $x = 9.8 - \frac{1}{2}(9.8)\,(1) = 4.9$

$\therefore$ The maximum height attained by the particle is 4·9 m.

Ex. 4. A body is released from a height of 490 m. Find the time taken by the body to reach the ground. How much distance does it cover in the 4th second of its motion.

Sol Let x be the distance of the body vertically below the point of release at time t. Then the differential equation of its motion is

$$\frac{d^2x}{dt^2} = g$$

$\therefore \quad \int \frac{d^2x}{dt^2}\, dt = \int g\, dt + c_1$

$\therefore \quad \frac{dx}{dt} = gt + c_1$

i.e. $\quad v = gt + c_1 \qquad (2)$

Since the body starts from rest. $v = 0$, when $t = 0$

$\therefore \quad 0 = g\,(0) + c_1 \quad i.e.\ c_1 = 0.$

$\therefore \quad v = gt.$

i. e. $\frac{dx}{dt} = gt$

$\therefore \quad dx = (gt)\, dt$

$\therefore \quad \int dx = \int gt\, dt + c_2$

$\therefore \quad x = \frac{1}{2} gt^2 + c_2$

Since x is measured from the point of release, $x = 0$ when $t = 0$.

$\therefore \quad 0 = \frac{1}{2} g\,(0)^2 + c_2 \qquad i.e.\ c_2 = 0$

$\therefore \quad x = \frac{1}{2} gt^2$

When the body reaches the ground $x = 490$ m

$\therefore \quad 490 = \frac{1}{2}(9.8)t^2 \qquad$ (taking $g = 9.8$ m/sec^2)

$\therefore \quad t^2 = \frac{2 \times 490}{9.8} = 100$

$\therefore \quad t = 10.$ *i.e.* the body takes 10 seconds to reach the ground.

Now at the end of 3 seconds, the distance covered is

$$x_1 = \frac{1}{2} g\,(3)^2 = \frac{9}{2} g$$

At the end of 4 seconds the distance covered is,

$$x_2 = \tfrac{1}{2}\, g\, (4)^2 = \frac{16}{2} g$$

i.e. $x_2 = 8g$

∴ The distance covered by the body in the 4th second is,

$$x_2 - x_1 = 8\, g - \frac{9}{2} g$$

$$= \frac{7}{2} g$$

$$= 34{\cdot}3 \text{ m taking } g = 9{\cdot}8 \text{ m/sec}^2.$$

∴ The distance covered by the body in the 4th second is 34·3 m.

7·12. Simple Harmonic Motion. (S.H.M.)

A particle is said to execute simple harmonic motion if it moves along a straight line in such a way that its acceleration is always directed towards a fixed point in that line and is proportional to the distance of the particle from that point. The fixed point is called the centre of the S.H.M.

Let us obtain the differential equation of a S H.M.

Let the particle move along $X'X$ and let the fixed point be O, the origin.

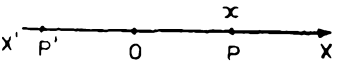

Fig. 7·12

Let x be the distance of the particle from O at time t.

∴ The acceleration of the particle at time t is $\dfrac{d^2x}{dt^2}$

By the definition of the S.H.M. this is proportional to x and is directed towards O.

$$\therefore \quad \frac{d^2x}{dt^2} = -k^2x \qquad 1$$

k^2 is the constant of proportionality. It is taken in the form k^2 where k is a real number to ensure that the constant of proportionality is positive. The negative sign on the r.h·s. ensures that the acceleration is always directed towards O. Thus if $x>0$ *i.e.* the particle is at some point P on the r.h.s. of O, the acceleration is −ve and hence in the direction PO.

If the particle is at P' a point on the l.h.s. of O, the $x<0$ and the acceleration is positive and hence in the direction P'O. In both cases it is directed towards O.

Equation (1) is the differential equation of the S.H.M. It can also be written as,

$$\frac{d^2x}{dt^2} + k^2x = 0$$

7·13. Solution of $\frac{d^2x}{dt^2} + k^2x = 0$

We write the equation as,

$$\frac{d^2x}{dt^2} = -k^2 x \qquad \ldots 1$$

Now $\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$

$$= \frac{dv}{dt}$$

and $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$

$$= \frac{dv}{dx} v$$

$\therefore \quad \frac{d^2x}{dt^2} = v \frac{dv}{dx}$

$\therefore$ (1) gives $v\frac{dv}{dx} = -k^2x$

i.e. $v\,dv = -k^2x\,dx$

Integrating both sides

$\frac{1}{2} v^2 = -\frac{1}{2}k^2x^2 + c_1$

$\therefore \quad v^2 = c - k^2x^2$ where $c = 2c_1$. $\qquad \ldots 2$

since $v = \frac{dx}{dt}$ we have

$$\left(\frac{dx}{dt}\right)^2 = c - k^2x^2$$

$$\therefore \frac{dx}{dt} = \sqrt{c - k^2x^2} \qquad \ldots 3$$

$$= k\sqrt{\frac{c}{k^2} - x^2}$$

$$\therefore \frac{dx}{\sqrt{\frac{c}{k^2} - x^2}} = k\,dt$$

Integrating both sides we get,

$\sin^{-1} \frac{x}{\sqrt{c/k^2}} = kt + \alpha$, α being the constant of integration.

$$\therefore x = \sqrt{\frac{c}{k^2}} \sin(kt + \alpha)$$

Writing $a = \sqrt{\frac{c}{k^2}}$ we get,

$$x = a \sin(kt + \alpha) \qquad \ldots 4$$

In (4) a and α are arbitrary constants and it is a solution of equation **1**. Since (4) contains two arbitrary constants it is the general solution of (1), the differential equation of SHM.

Remarks. (1) Positive sign is attached to the velocity in equation (3). If we take the negative sign, then the integral in the next step will be,

$$\cos^{-1}\left(\frac{x}{\sqrt{c/k^2}}\right) = kt + \beta$$

$$\therefore \quad x = \sqrt{\frac{c}{k^2}} \cos(kt + \beta)$$

$$= \sqrt{\frac{c}{k^2}} \sin\left(kt + (\beta + \frac{\pi}{2})\right)$$

Writing $\alpha = \beta + \frac{\pi}{2}$, we get

$$x = \sqrt{\frac{c}{k^2}} \sin(kt + \alpha) \text{ as before.}$$

$\therefore$ the general solution may be taken as

$x = a \sin(kt + \alpha)$ or $x = a \cos(kt + \alpha)$. These are two equivalent forms of the general solution. Further we may write

$$x = a \sin(kt + \alpha)$$
$$= a(\sin kt \cos\alpha + \cos kt \sin x).$$
$$= (a \cos\alpha) \sin kt + (a \sin\alpha) \cos kt.$$

Writing $c_1 = a \cos\alpha$ and $c_2 = a \sin\alpha$, the general solution can also be written as,

$$x = c_1 \sin kt + c_2 \cos kt.$$

2. Since $-1 \leqslant \sin(kt + \alpha) \leqslant 1$

$$\therefore \quad -a \leqslant a \sin(kt + \alpha) \leqslant a$$

$$\therefore \quad -a \leqslant x \leqslant a$$

$\therefore$ The particle cannot go beyond a distance a on either side of the origin. This distance a is called the *amplitude* of the SHM.

3. $x = a \sin(kt + \alpha)$

$$\therefore \quad \frac{dx}{dt} = ak \cos(kt + \alpha)$$

$\therefore$ We have $x = a \sin(kt + \alpha)$ and $v = ak \cos(kt + \alpha)$.

Now $\sin(kt + \alpha)$ and $\cos(kt+\alpha)$ are both periodic functions of t with period $\frac{2\pi}{k^2}$.

$\therefore$ the particle acquires the same position, the same velocity and the same acceleration after every interval of time $\frac{2\pi}{k}$. Hence

the motion of the particle is said to be periodic and the time interval $\frac{2\pi}{k}$ is called the period of motion. This is denoted by T. Thus $T = \frac{2\pi}{k}$.

4. The movement of the particle from a position P till it returns to P again is called an oscillation of the motion. Thus T is the time taken for one oscillation. The number of oscillation in a unit of time is called the frequency of the SHM. The frequency is denoted by n.

$$\therefore \quad n = \frac{1}{T} = \frac{k}{2\pi}$$

5. We have,

$$v = ak \cos(kt+\alpha)$$
$$= k\sqrt{a^2 - a^2 \sin^2(kt+\alpha)}.$$
$$\therefore \quad v^2 = k^2(a^2 - x^2)$$

$\therefore$ v^2 *i.e.* v is maximum when $x=0$ and the maximum velocity is $v=ka$. This is when the particle is at the origin.

Also when $x = \pm a$ *i.e.* when the particle is at the extreme position.

$$v^2 = k^2(a^2 - a^2) = 0$$

i.e. $v = 0$.

7·14. Particular Forms.

We shall now obtain two special forms of the equation of SHM. The general equation is,

$$x = a \sin(kt+\alpha)$$

where a and α are parameters.

(*i*) Suppose the particle starts at $x = p$ with velocity $v = 0$. These are called initial conditions of the motion.

We have,

$x = a \sin(kt + \alpha)$ and $v^2 = k^2(a^2 - x^2)$

At $t=0$,

$p = a \sin\alpha$ and $0 = k^2(a - p^2)$

$\therefore$ $p = p \sin\alpha$ *i.e.* $\sin\alpha = 1$

$\therefore$ $\alpha = \frac{\pi}{2}$

$\therefore$ $x = p \sin\left(kt + \frac{\pi}{2}\right)$ *i.e.* $x = p \cos kt$

is the equation of the SHM.

(*ii*) Suppose the particle starts from the origin with velocity v_0

$\therefore$ at $t=0$, $x=0$ and $v=v_0$

We have,

$x = a \sin (kt + \alpha)$ and $u^2 = k^2 (a^2 - x^2)$

$\therefore \quad 0 = a \sin \alpha$ and $v_0^2 = k^2 a^2$

Since $a \pm 0 \sin \alpha = 0$ *i.e.* $\alpha = 0$ and from the second equation we get $a = \dfrac{v_0}{\sqrt{k}}$

$\therefore$ The equation of the SHM is,

$$x = \frac{v_0}{\sqrt{k}} \sin kt.$$

Thus when the particle is in an extreme position when $t = 0$, the equation of the motion is,

$$x = a \cos kt$$

and if it is in the centre when $t = 0$, the equation is,

$$x = a \sin k t,$$

a being the amplitude of the motion.

Illustrative examples :

Ex. 1. A particle executes SHM along the axis of x with period π, When $t = 0$, $x = 2$ and $v = 4$. Find the amplitude of the motion.

Sol. Let the equation of the motion be,

$$x = a \sin (kt + \alpha)$$

$\therefore \quad v = ak \cos (kt + \alpha)$

Now $T = \dfrac{2\pi}{k} = \pi \quad \therefore k = 2$

Also $x = 2$ and $v = 4$ when $t = 0$

$\therefore \quad 2 = a \sin \alpha$ and $4 = a\, 2 \cos \alpha$

i.e. $2 = a \cos \alpha$

$\therefore \quad a^2 \sin^2 \alpha + a^2 \cos^2 \alpha = 4 + 4 = 8$

i.e. $a^2 = 8 \quad \therefore a = 2\sqrt{2}$ is the amplitude of the motion.

Ex. 2. The position of a particle moving along the axis of x at time t is given by,

$$x = 3\cos 2t + 4 \sin 2 t$$

Form the differential equation of the motion. Hence or other wise show that it is SHM. Find its amplitude and period.

Sol. We have,

$$x = 3 \cos 2 t + 4 \sin 2 t$$

$$\therefore \frac{dx}{dt} = -6 \sin 2 t + 8 \cos 2 t$$

Differentiating again,

$$\frac{d^2x}{dt^2} = -12\cos 2t - 16\sin 2t$$

$$= -4(3\cos 2t + \sin 2t)$$

$$= -4x$$

$\therefore$ The differential equation is,

$$\frac{d^2x}{dt^2} + 4x = 0$$

Comparing with the equation of the general SHM *e.g.*

$$\frac{d^2x}{dt^2} + k^2x = 0$$

We see that, the motion is SHM with $k=2$.

$\therefore$ the period of the motion is $T = \frac{2\pi}{k} = \pi$. $\quad \because k = 2$

Now $x = 3\cos 2t + 4\sin 2t$

$$= 5\left[\frac{3}{5}\cos 2t + \frac{4}{5}\sin 2t\right]$$

Let α be such that,

$$\tan\alpha = \frac{3}{4}$$

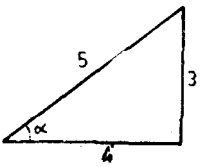

Fig. 7.13

$\therefore \quad \sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$

$\therefore \quad x = 5[\sin\alpha\cos 2t + \cos\alpha\sin 2t]$

$= 5\sin(2t + \alpha)$.

This is the equation of a SHM with amplitude 5 and period π.

Ex. 3. A particle executes SHM along the axis of x with amplitude 6 and period $\frac{\pi}{4}$. With what velocity does it pass the point $x=-3$.

Sol. Let the equation of motion be

$$x = a\sin(kt + \alpha)$$

Since the amplitude is 6, $a = 6$.

Also the period is $\frac{\pi}{4}$ $\therefore \frac{2\pi}{k} = \frac{\pi}{4}$

$\therefore \quad k = 8$

$\therefore$ The equation of the motion is,

$$x = 6\sin(8t + \alpha)$$

Now $v^2 = k^2(a^2 - x^2)$

$= 8^2(6^2 - x^2)$

$\therefore$ when $x = -3$,

$v^2 = 8^2(6^2 - 9)$

$= 8^2 \times 27$

$\therefore$ The particle passes the point $x = -3$ with velocity $\pm 24\sqrt{3}$.

Ex. 4. A particle describing SHM executes 120 oscillations per minute and its velocity at the centre is 44 m/s. What is the amplitude and what is the velocity when it is half way between the centre and an extreme position $\left(\text{take } \pi = \frac{22}{7}.\right)$

Sol. 120 oscillations per minute means 2 oscillations per second.

$$\therefore \quad \text{The period } T = \frac{1}{\text{frequency}} = \frac{1}{2}.$$

$$\therefore \frac{2\pi}{k} = \frac{1}{2}$$

$$\therefore \quad k = 4\pi = \frac{88}{7}$$

The velocity at the centre is the maximum velocity ak

$$\therefore a.\frac{88}{7} = 44$$

$$\therefore \quad a = \frac{7}{2} = 3{\cdot}5 \text{ meters.}$$

Also $v^2 = k^2(a^2 - x^2)$

$\therefore$ When $x = \frac{a}{2}$,

$$v^2 = k^2\left(a^2 - \frac{a^2}{4}\right)$$

$$= k^2\frac{3a^2}{4}$$

$$\therefore \quad v = \pm \frac{ka\sqrt{3}}{2}$$

$$= \pm\frac{88}{7}.\frac{7}{2}\frac{\sqrt{3}}{2}$$

$$= \pm 22\sqrt{3} \text{ m/s.}$$

Ex 5. The coordinate of a particle moving along the axis of x at time t is given by $x = 2\sin^2 3t$. Show that the motion is simple harmonic about the point $x = 1$ as centre. Show further that the period is $\frac{\pi}{3}$ and find the amplitude.

Sol. We have,

$$x = 2\sin^2 3t$$

$$= 1 - \cos 6t$$

$$\therefore \quad x - 1 = -\cos 6t$$

$\therefore$ Shifting the origin to $x = 1$, $X = x - 1$

$$X = -\cos 6t$$

and $$\frac{d^2X}{dt^2} = 36 \cos 6t$$

i.e. $$\frac{d^2X}{dt^2} = -36X$$

This is the differential equation of a SHM.

$\therefore$ the motion is simple harmonic with $X = 0$ as the centre *i.e.* $x = 1$ as the centre

Also $k = 6$

$\therefore$ The period of the motion is,

$$T = \frac{2\pi}{k} = \frac{2\pi}{6} = \frac{\pi}{3}.$$

Since $X = -\cos 6t$, max $X = 1$

$\therefore$ The amplitude of the motion is 1.

Ex. 6. A particle describes SHM of amplitude 4 cms. If its acceleration at a distance of 1 cm from the mean position is 3 cms/sec^2, find its velocity at a distance of 2 cms. from the mean position.

Sol. The equation of motion is,

$$\frac{d^2x}{dt^2} = -k^2x$$

when $x = 1$, the acceleration is -3,

$\therefore$ $-3 = -k^2(1)$

$\therefore$ $k = \sqrt{3}$

Also $v^2 = k^2(a^2 - x^2) = 3(16 - x^2)$

The velocity at a distance of 2 cms from the mean position *i.e.* when $x = 2$ will be,

$$v^2 = 3(16 - 4) = 36$$

$\therefore$ $v = \pm 6$ cms/sec.

Ex. 7. A particle is moving along the axis of x. Its distance x from the origin at time t is given by $x = 2 + 3\cos 2t + 4\sin 2t$. Show that it executes SHM. Find the period and amplitude.

Sol. We have $x = 2 + 3\cos 2t + 4\sin 2t$

i.e. $x - 2 = 3\cos 2t + 4\sin 2t$

Shifting the origin to $x = 2$ $\quad X = x - 2$

$$X = 3\cos 2t + 4\sin 2t$$

$\therefore$ $$\frac{d^2X}{dt^2} = -4(3\cos 2t + 4\sin 2t)$$

$$= -4X.$$

$\therefore$ the particle executes SHM with $X = 0$ as the centre *i. e.* with $x = 2$ as the centre

Also $k^2 = 4$ *i.e.* $k = 2$

$\therefore$ the period of the motion $T = \frac{2\pi}{2} = \pi$

we have $X = 3 \cos 2t + 4 \sin 2t.$

$$= 5\left[\frac{3}{5}\cos 2t + \frac{4}{5}\sin 2t\right]$$

Let α be such that $\tan\alpha = \frac{3}{4}$

$\therefore \sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$

$\therefore X = 5(\sin\alpha\cos 2t + \cos\alpha \sin 2t)$

$= 5 \sin(2t + \alpha)$

$\therefore$ the amplitude is 5.

EXERCISE 7·3

1. A falling body passes through 224 ft in the last second of its fall. Find the height from which it fell and the time of its falling.

2. A tower is 260 ft high and a body is projected from its base with a velocity of 128 ft. sec. After one second another body is let fall from the top of the tower. At what height will they meet ?

3. A stone is let fall from a height of 240 ft, and at the same time a stone is projected upwards from the ground to meet it. With what velocity must the second stone be projected in order that the stones may meet at a height of 96 ft.

4. A stone is projected vertically upwards with a velocity of 30 ft/sec. Two seconds later another stone is projected from the same point with the same velocity. Find when and where they will meet.

5. A stone thrown vertically upwards passes a certain point in t_1 seconds after its projection and after a further t_2 seconds strikes the ground. Prove that the height of this point is gt_1t_2. Prove further that the stone passes a point midway between this point and the point of projection with a velocity $\frac{1}{2} g \sqrt{t_1^2 + t_2^2}$.

6. A pa ticle is projected vertically upwards. Show that if t_1, t_2 are the times at which it passes a point at a height h above the point of projection in assending and descending respectively, then,

$$h = \frac{1}{2} gt_1t_2$$

7. A particle executes SHM of period π secs. If its maximum acceleration is 5 meters/sec^2, find the amplitude and the velocity when the particle is at a distance of one meter from the centre of the SHM.

8. A particle executing a SHM acquires velocities of 13 metres per sec and 5 metres per sec at distances 3 metres and 5 metres respectively from the centre. Find the amplitude and the periodic time of the motion.

9. A particle executing SHM passes the centre with a velocity which is twice its velocity as it passes through a point at a distance of 5 from the the centre. Find the amplitude of the motion.

10. The period of a SHM is T. At a distance x_0 from the centre the velocity is V_o. Find the amplitude of the motion.

11. The amplitude of a SHM is 4. As the particle passes through the centre the velocity is $\pm$ 5. Find the period of the motion.

12. If x is the x-coordinate of a particle moving along the x-axis. at time t show that the motion given by each of the following equations is simple harmonic :

(*i*) $x = 5\cos 2t + 12\sin 2t$

(*ii*) $x = A\cos pt + B\sin pt$

(*iii*) $x = 2\sin 3t - \frac{1}{2}\cos 3t + 1$

(*iv*) $x = 3\cos t + 4\sin t + 3$.

Find the amplitude and the periodic time of each of these motions.

13. A particle executing SHM has velocities 5 cm./sec. and 4 cms-/sec. When it is at a distance of 2 cms. and 3 cms. respectively from the centre, find the period and the amplitude.

14. A particle performing SHM starts from a point 14 metres distant from the centre. If its maximum velocity is 22 metres/sec. find its periodic time.

15. A particle excuting a SHM of amplitude 16 metres takes 6 secs to travel 8 $\sqrt{3}$ metres from the centre. Find the period.

16. A particle executes SHM with period π. Find the equation of motion if at $t=0$, the distance of the particle from the origin is 1 and it has velocity 2 at that time.

17. A particle executes a SHM of amplitude 4 metres. When it is half way between the centre and an extremity its speed is 12 $\sqrt{3}$ metres/sec. What is the period of the motion ?

18. A body performing SHM has amplitude a and period T. Show that the velocity V at a distance X from the mean position is given by

$$V^2T^2 = 4\pi^2(a^2 - x^2).$$

19. A body is projected vertically upwards with a velocity 98 m./sec. How high will it rise and how much time will it take to return to the point from which it is projected ?

20. A stone is released from a tower 176.4 m in height, Find the velocity with which it will strike the ground. What is the distance covered by it in the last second before striking the ground ?

21. A stone is released from a height of 490 metres. Find the time it takes to reach the ground. What is the distance covered by it in the 4th sec of its motion.

22. A body fallen from the top of a tower covers 53·9 metres in the last second of its motion. Find the height of the tower.

ANSWERS

1. 7·5 secs., 900 ft ; **2.** $221\frac{35}{36}$ ft ; **3.** 80 ft/sec. ;

4. 3·5 secs., **5.** 84 ft. ; **7.** $\frac{5}{4}$ metres ; $\frac{3}{2}$ metres/sec ;

8. $\frac{5}{3}\sqrt{10}$ metres; $\frac{2\pi}{3}$ secs ; **9.** $\frac{10}{3}\sqrt{3}$

10. $\sqrt{x_0^2 + \frac{v_0^2 T^2}{4\pi^2}}$; **11.** $\frac{8\pi}{5}$; **12.** (*i*) $13, \pi$;

(*ii*) $\sqrt{A^2+B^2}$; $\frac{2\pi}{p}$; (*iii*) $\frac{\sqrt{17}}{2}$, $\frac{2\pi}{3}$; (*iv*) 5; 2π ;

13. 4.7 secs. ; 4.22 cms. **14.** 4 secs. ; **15.** 36 secs. ;

16. $x = \sqrt{2} \sin\left(2t + \frac{\pi}{4}\right)$ **17.** $\frac{\pi}{3}$ **19.** 490 m ; 20 secs ;

20. 53·9 m. 58·8 m/sec. **21.** 10 sec ; 34·3 m **22.** 176·4 m.

7·15. Problems on growth and decay.

Many measurable characteristics in daily life, such as mass of a radioactive substance, temperature of a body, population of a country etc., grow or decay with time. We can consider their rates of change *w.r.t* time. It x represents a measure of the characteristic $\frac{dx}{dt}$ will represent the rate of change of the characteristic *w. r. t.* time. If this rate of change is some known function we can write down a differential equation satisfied by x as a function of t. The solution of this differential equation will then determine x as function of t. This will be illustrated by the following examples.

Ex. 1. The rate of growth of the population of a country at any time is proportional to the size of the population at that time. For a certain country it is found that the constant of proportionality is 0·04. Show that the population of the country will be more than doubled in 25 years.

Sol. Let x = population of the country after t years.
from a certain fixed instant say today.
P = population of the country at the fixed instant.

$\therefore \quad \frac{dx}{dt} \propto x$ and since the constant of proportionality is 0·04,

$$\frac{dx}{dt} = (0{\cdot}04)\, x.$$

This is the differential equation satisfied by x.

$$\therefore \frac{dx}{x} = (0{\cdot}04)\, dt.$$

Integrating we get,

$$\log x = (0{\cdot}04)\, t + c.$$

$$\therefore \quad x = e^{(0{\cdot}04)t + c}$$

$$= e^c \cdot e^{(0{\cdot}04)t}$$

when $t = 0$ *i. e.* at the fixed instant, $x = P$.

$$\therefore \quad P = e^c \cdot e^{(0{\cdot}04)(0)}$$

$$= e^c .$$

$\therefore \quad x = P\ e^{(0\cdot04)t}$.

$\therefore$ after 25 years *i.e* at $t=25$,

$$x = P\ e^{(0\cdot04)\ 25}$$

i.e $x = P.\ e$

$\therefore\ x > 2P \quad$ since $e > 2$.

$\therefore$ the population will be more than doubled in 25 years.

Ex. 2. The rate of decay of the mass of a radioactive substance at any instant is α times its mass at that instant. It $\alpha = 10^{-4}$, show that the mass of the substance will be less than half its value today after 10,000 years.

Sol. Let x = mass of the substance after t years from today.

m = its mass today.

Since the mass decays *i.e.* reduces with time its rate of change *w.r.t.* time is negative,

$$\therefore \frac{dx}{dt} = -(10^{-4})\ x.$$

$$\therefore \frac{dx}{x} = -(10^{-4})\ dt.$$

Integrating we get,

$$x \log = c-(10^{-4})\ t$$

$$\therefore \quad x = e^{c-(10^{-4})\ t}$$

$$= e^{c}\ .\ e^{-(10^{-4})\ t}$$

When $t = 0$ *i.e* today, the mass is m

$$\therefore \quad m = e^{c}\ .\ e^{-(10^{-4})\ (0)}$$

$$= e^{c}$$

$$\therefore \quad x = m.\ e^{-(10^{-4})\ t}$$

After 10,000 years *i.e.* at $t = 10{,}000$,

$$x = m\ e^{-(10^{-4})\ \cdot\ 10000}$$

$$= me^{-1}$$

$\therefore \quad x < \frac{1}{2}\ m \qquad \therefore \quad e > 2 \Rightarrow e^{-1} < \frac{1}{2}$.

$\therefore$ the mass will be less than half the mass today, after 10,000 years.

Ex. 3. The rate of decay of the mass of a radioactive substance at an instant is proportional to its mass at that instant. It m_0 is the initial mass of the substance find the expression for its mass at time t. If it is known that its mass in 100 years becomes $\frac{4}{5}$th its initial mass, what will be its mass after 500 years.

Sol. Let m denote the mass of the substance at time t. Then $\frac{dm}{dt}$ is the rate of change of its mass *w.r.t.* time t.

$$\therefore \quad \frac{dm}{dt} \propto m$$

Since the mass decays i.e. decreases, $\frac{dm}{dt}$ is negative,

$$\therefore \quad \frac{dm}{dt} = -km \qquad k>0.$$

k being the constant of proportionality.

$$\therefore \quad \frac{dm}{m} = -kdt.$$

Integrating we get,

$$\log m = -kt + c$$

$$\therefore \quad m = e^{c-kt}$$

$$= e^{c} \cdot e^{-kt}$$

At $t = 0$, $m = m_0$,

$$\therefore \quad mo = e^{c} \cdot e^{-k(0)}$$

$$= e^{c}$$

$$\therefore \quad m = m_0 \, e^{-kt}.$$

This gives the mass of the substance at time t.

Now at $t = 100$, $m = \frac{4}{5} m_0$

$$\therefore \quad \frac{4}{5} m_0 = m_0 e^{-k \cdot 100}$$

$$\therefore \quad e^{-100k} = \frac{4}{5}.$$

After 500 years *i.e.* at $t = 500$, the mass will be,

$$m = m_0 \cdot e^{-k500}$$

$$= m_0 \left(e^{-100k}\right)^5$$

$$= m_0 \left(\frac{4}{5}\right)^5$$

$\therefore$ the mass after 500 years will be,

$$m_0 \left(\frac{4}{5}\right)^5 = (\cdot 33)\, m_0 \text{ approximately.}$$

7.16. Newton's Law of Cooling

When any body is at a higher temperature than its surroundings and is left to itself *i.e.* no heat is supplied to it, it tends to cool down to the temperature around it. Newton postulated the following law regarding the rate of loss of heat by the body :

The rate of loss of heat of a body is proportional to the difference between its temperature and the ambient temperature i.e. the temperature of its surroundings.''

Let Q be the quantity of heat contained in the body at temperature T°C. Let T_0°C be the temperature of its surroundings.

$\therefore$ the rate of change of heat Q w.r.t. time t_1

is $\frac{dQ}{dt}$. This is proportional to T—To.

$\therefore \frac{dQ}{dt} \propto$ T—To

Since heat is being lost $\frac{dQ}{dt}$ is negative, and we have

$$\frac{dQ}{dt} = -k\,(T-T_0),\ k>0 \qquad 1$$

k being the constant of proportionality.

Now Q the amount of heat contained in a body when its temperature is T°C is,

$$Q = m.s.\,(T-T_0) \qquad 2$$

where m = mass of the body

s = its specific heat.

$$\therefore \frac{dQ}{dt} = m.s.\,\frac{dT}{dt} \qquad 3$$

From (1) and (3) we obtain,

$$m.s.\,\frac{dT}{dt} = -k\,(T-T_0)$$

$$\therefore m.s.\,\frac{dT}{dt} = -k\,(T-T_0)$$

$$i.e.\ m.\,s.\,\frac{dT}{T-T_0} = -k\,d\,t$$

Since m, the mass and s the specific heat are constants, we get on integration,

$$m.s.\log\,(T-T_0) = c-kt$$

$$i.e.\ \log\,(T-T_0) = \frac{c}{m.s.} - \frac{kt}{m.s.}$$

$$\therefore\ T - T_0 = e^{\frac{c}{m.s.} - \frac{kt}{m.s.}}$$

If at $t=0$, $T-T_0 = \alpha$ *i.e.* the difference of temperature between the body and its surroundings initially was α,

$$\alpha = e^{\frac{c}{m.s.} - \frac{k(0)}{m.s.}}$$

$$= e^{\frac{c}{m.s.}}$$

$\therefore$ (4) gives,

$$T - T_0 = \alpha\, e^{\frac{-kt}{m.s.}}$$

$$\therefore \quad T = T_0 + \alpha\, e^{\frac{-kt}{m.s.}}$$

This will be the temperature of the body after a time interval t.

Illustrative Examples

Ex. 1. A large brass utensil of mass m at a temperature of 400°C. cools in an atmospheric temperature of 25°C according to Newton's law. If the specific heat of brass is 0·1, write down the expression for its temperature after time t. If in 20 minutes its temperature becomes 325°C what will be its temperature after one hour ?

Sol. We have,

$$T = T_0 + \alpha e^{\frac{-kt}{ms}}$$

where T = temperature at time t

T_0 = atmospheric temperature = 25°C

α = $T - T_0$ initally = 400° − 25° = 375°

m = mass of the body

s = Specific heat of the substance = 0·1.

$\therefore$ for the brass utensil,

$$T = 25 + 375\, e^{\frac{-kt}{m(0\cdot1)}}$$

$$\therefore T - 25 = 375.e^{\frac{-kt}{m(0\cdot1)}}$$

This determines the temperature of the utensil at time t.

After 20 minutes $T = 325°$

$$\therefore 325 - 25 = 375\, e^{\frac{-k\,(20)}{m\,(0.1)}}$$

$$\therefore e^{\frac{-200k}{m}} = \frac{300}{375}$$

$$= \frac{4}{5}$$

$\therefore$ the temperature after one hour *i.e.* 60 minutes will be given by,

$$T - 25 = 375\, e^{\frac{-k(60)}{m\,(0{\cdot}1)}}$$

$$= 375 \left(e^{\frac{-200k}{m}} \right)^3$$

$$= 375 \left(\frac{4}{5}\right)^3$$

$$= 192$$

$\therefore$ T = 217°C at the end of one hour.

Ex. 2. A body of mass 500 gms. and specific heat 0·5, cools according to Newtons' law from a temperature of 150°C in an atmosphere of 30°C. Obtain an expression for its temperature at time t. If the body cools to 90° in 30 minutes, what will be the time taken by it to cool down to 60°.

Sol. Newtons' law states that the rate of loss of heat is proportional to the difference of temperature between the body and its surroundings.

$\therefore$ denoting by Q the quantity of heat contained in the body at time t and by T, T_0, the temperature of the body and that of its surroundings respectively, we have,

$$\frac{dQ}{dt} \propto T - T_0$$

As heat is being lost $\frac{dQ}{dt}$ is negative ;

$$\therefore \quad \frac{dQ}{dt} = -k(T - T_0) \qquad k > 0$$

$$= -k(T - 30°) \qquad \text{Since } T_0 = 30° \qquad (1)$$

Also $Q = ms\,(T+273)$,

m being the mass and s the specific heat of the body.

$$\therefore Q = 500 \times 0.5\,(T+273)$$

$$= 250\,(T+273)$$

$$\therefore \frac{dQ}{dt} = 250\,\frac{dT}{dt} \qquad (2)$$

(1) and (2) gives

$$250\,\frac{dT}{dt} = -k\,(T-30)$$

$$\therefore \frac{dT}{T-30} = -\frac{k}{250}\,dt$$

Integrating we obtain,

$$\log (T-30) = -\frac{k}{250}\,t + c$$

$$\therefore \quad T - 30 = e^{c - \frac{k}{250} t}$$

when t = 0, T = 150°

$$\therefore \quad 150 - 30 = e^{c - \frac{k}{250}(0)}$$

$$= e^{c}$$

$$\therefore \quad e^{c} = 120$$

$$\therefore \quad T - 30 = 120 \cdot e^{-\frac{k}{250} t}$$

$$i.e. \quad T = 30 + 120 \cdot e^{-\frac{k}{250} t}$$

This relation gives the temperatare of the body at time t.

Now T = 90° after 30 minutes

$$\therefore \quad 90 - 30 = 120 \cdot e^{\frac{-k}{250}(30)}$$

$$i.e. \quad 60 = 120\, e^{-\frac{30k}{250}}$$

$$\therefore \quad e^{-\frac{30k}{250}} = \tfrac{1}{2} \qquad \ldots\ldots\ldots 3$$

If the temperatare is 60° at $t = t_1$,

$$60 - 30 = 120\, e^{-\frac{kt_1}{250}}$$

$$i.e. \quad 30 = 120\, e^{-\frac{kt_1}{250}}$$

$$\therefore \quad e^{-\frac{kt_1}{250}} = \tfrac{1}{4} \qquad \ldots\ldots\ldots 4$$

From (3) we get

$$\left(e^{-\frac{30k}{250}} \right)^2 = \tfrac{1}{4}$$

$$\text{i. e.} \quad e^{-\frac{60k}{250}} = \tfrac{1}{4} \qquad \ldots\ldots\ldots 5$$

From (4) and (5), t_1 = 60 minutes

$\therefore$ the body will cool down to 60° in one hour.

EXERCISE 7.4.

1. The rate of increase of the mass of a water drop by condensation from surrounding air is proportional to its mass at the time. Find an expression for its mass at time t if the initial mass is m_0. If in 20 seconds its mass increases by $\frac{9}{10} m_0$, What will be the increase in mass in the next 30 seconds.

2. The rate of increase of the population of a country at any time is proportional to the size of the population at that time. Find an expression for the size of the population after t years from today if the population today is P. The population of a country increases by half its value today within a period of next 10 years. What will be its population 30 years from now ?

3. The mass of a crystalline deposit increases at a rate which is proportional to its mass at that time. The deposit has started around a crystal seed of 5 gms. Find an expression for its mass at time t. If in 30 minutes the mass of the deposit increases by 1 gm. what will be the mass of the deposit after 10 hours.

4. The rate of decay of the mass of a radio active substance is proportional to its mass at that instant. Find an expression for its mass ofter t years if today it is m_0. Existing evidence suggests that it will loose $\frac{1}{10}$ th of its mass in 100 years. What will be its mass after 1000 years ?

5. The rate of decay of a self destroying bacterial culture is proportional to $\frac{1}{2}$ the number of live cells of the culture at the time. Find an expression for the mass of live cells at time t if initially it is m_0. If half the number is destroyed in 60 minutes what will be number of live cells after 4 hours.

6. It is found that the rate of decrease of the mass of a volatile liquid is proportional to $(t+1)^{-2}$ at t seconds, from the instant of exposure to atmosphere. Show that the mass left in the container after t seconds is $m_0 - \frac{kt}{t+1}$, where k is the constant of proportionality and m_0 the mass at $t=0$. What will bs the time taken by the liquid to evaporate completely ?

7. The fixed deposits with a branch of certain bank are increasing at the rate of $6t-a$ at the end of time t from the date the branch is opened. Show that the deposits at time t are given by- $3t^2-at$.

8. A body cools according to Newtons' law from 100° to 60° in 20 minutes. The temperature of the surroundings is 20°C. How long will the body take to cool down to 30° ?

9. A mass m gms, of boiling water is left to cool in atmospheric temperature of 25°C, If it cools according to Newton's law find an expression for its temperature at the end of t seconds (Boiling temperature of water is 100°C and specific heat of water is 1). If it cools down from 100° to 40° in 30 minutes what will be its temperature after one hour ?

10. A silver plate of mass 500 gms was heated to 200°C and left to cool in an atmosphere at a temperature of 25°C. If the specific heat of silver is 0.056 and the cooling is according to Newton's law, find an expression for

the temperature of the plate after time t. If after 20 minutes its temperature is 150°C what will be its temperature after one hour ?

11. A copper tank at a temperature of 250°C is allowed to cool in a temperature of 50°C, If it cools according to Newton's law and the specific heat of copper is 0·1, find an expression for its temperature at time t. If after 40 minutes its temperature is 200°C what will be its temperature at the end of 2 hours ?

12. Population increases at a rate which is proportional to the size of the population at the time. The population of a city increases from 20 lakhs to 40 lakhs in 30 years. Find the population after another 15 years.

13. The rate of growth of bacterial culture is proportional to the number of live cells at the time. Initially the number of live cells was 1000 and after 10 hours their number was 8000. Find the number of live cells 15 hours from the beginning.

14. A body cooling according to Newton's law cools down from 80°C to 60°C in 2 minutes. If the temperature of the surroundings is 10°C in how much time it will cool from 60°C to 40°C.

15. The temperature of a liquid is 75°C when the room temperature is 25°C. After 15 minutes the temperature of the liquid is found to be 65°C. Show that after 90 minutes from the initial observation the temperature of the liquid is 38°C.

ANSWERS

1. $\frac{25}{64}m_0$; $m_0e^{\alpha t}$; **2.** $\frac{27}{8}$ P ; $P_0e^{\alpha t}$; **3.** $5\left(\frac{6}{5}\right)^{20}$ gms ; $5e^{\alpha t}$;

4. $m_0e^{-\alpha t}$; $m_0\left(\frac{9}{10}\right)^{10}$; **5.** $\frac{81}{256}m_0$; $m_0e^{-\frac{1}{2}\alpha t}$ **8.** 60 min. ;

9. $T = 25 + 75\,e^{\frac{-kt}{m}}$; 28°C ; **10.** $T=25+175\,e^{\frac{-kt}{28}}$

11. $T=50+200\,e^{\frac{-10kt}{m}}$; $84\frac{3}{8}$° C. **12.** 56·4 lakhs ; **13.** 22624 ;

14. 5 minutes approximately.

Mathematics & Statistics Paper-II

Time : $2\frac{1}{2}$ *Hours* *Max. Marks* : *50*

Note : (*i*) All questions are compulsory.
(*ii*) All questions carry equal marks

1. Attempt any FIVE of the following :

(*i*) Find the g. l. b. and l. u. b. of the set

$$\{x \mid x^2-7x+12 < 0. \quad x \in R\}$$ 2

(*ii*) Discuss the convergence of the series

$$\sum \frac{1}{n^2+1}$$

(*iii*) Find $\frac{dy}{dx}$, if $\sqrt{x} + \sqrt{y} = a^2$ 2

(*iv*) Test the applicability of Lagrange's Mean Value theorem for

$$f(x) = 3+4x-x^2 \text{ in } [1, 4]$$ 2

(*v*) Evaluate $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ 2

(*vi*) Express $\int_0^3 (2x+3)\, dx$ as limit of a sum 2

(*vii*) Form differential equation whose solution is

$$Y = Ae^{2x}+Be^{-2x}$$ 2

(*viii*) Solve $(1+y)\, dx-(1-x)\, dy = 0$ 2

2. (*a*) Show that the sequence whose n^{th} term

$$a_n = 1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots\ldots+\frac{1}{n!}$$

is monotonic increasing and bounded. 4

(*b*) Answer any *three* of the following :

(*i*) Evaluate

$$\lim_{n\to\infty} [\sqrt{n^2+1}-n]$$ 2

(*ii*) Discuss convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$$ 2

(*iii*) Evaluate $\lim_{x \to 0} \frac{x \tan x}{1-\cos x}$ 2

(*iv*) Discuss continuity at $x = 4$ of the function

$$f(x) = \frac{x^2-16}{x-4} \quad \text{for } x \neq 4$$

$$= 9 \qquad \text{for } x = 4$$

(*v*) Obtain Derivative of $f(x) = \frac{1}{x+2}$

from the first principle. 2

3. (*a*) If u and v are derivable functions of x then show that

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad v \neq 0$$ 3

(*b*) If $x^p y^q = (x+y)^{p+q}$, then show that

$$\frac{dy}{dx} = \frac{y}{x}$$ 3

(*c*) (*i*) For the curve $y = be^{x/a}$ (where a and b are constants) show that subtangent is of constant length and sub-normal varies as square of the ordinate at any point of it. 2

(*ii*) The sides of a square are increasing at the rate of 0·5 cm./sec. At what rate is the area increasing when the side is 10 cm. long ? 2

OR

3. (*a*) A box with a square base is to have an open top. The area of the material for making the box is 192 sq. cms. What should be its dimensions in order that the volume is as large as possible ? 4

(*b*) Find the approximate value of $\tan^{-1}(0{\cdot}999)$ 2

(*c*) Evaluate

(*i*) $\lim_{\theta \to \pi/4} \frac{2-\operatorname{cosec}^2\theta}{1-\cos\theta}$ 2

(*ii*) If $y = x\sqrt{x}$ and $\frac{dy}{dx}$ 2

4. (*a*) If $f(x)$ and $g(x)$ are two integrable functions, show that $\int[f(x)+g(x)]\,dx = \int f(x)\,dx+\int g(x)\,dx$ and if k is constant then show that $\int k f(x)\,dx = k\int f(x)\,dx$. 3

(*b*) Integrate w. r. to x, any *two* of the following :

(*i*) $\int \frac{1}{(x+1)\sqrt{x-1}}\,dx$ 2

(*ii*) $\int \frac{x^2+2}{(x+2)(x+1)(x+3)}\,dx$ 2

(*iii*) $\int x^2 \log x\,dx$ 2

(*iv*) $\int \frac{\cos x}{9-\cos^2 x}\,dx$ 2

(*c*) A particle is projected vertically upwards with a velocity of 9·6 m/sec, Find how high it will rise ? 3

5. (*a*) Evaluate the following

(*i*) $\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$ 2

(*ii*) $\int_0^2 x\,e^x\,dx$ 2

(*b*) Find the area of the region bounded by the curve $y = x^2$ and the line $y = x$. 3

(*c*) If the bacteria in culture increase continuously at a rate proportional to the number present and initial number is N_0, find the number at time t. 3

OR

5. (*a*) If $f(x)$ is odd function *i.e.* $f(x) = -f(-x)$ then prove that

$$\int_{-a}^{a} f(x)\,dx = 0$$ 3

(*b*) Evaluate $\int_0^{\pi/2} \frac{\text{Sin}^2 x}{(1+\text{Cos } x)^2}\, dx$ 3

(*c*) (*i*) Solve the differential equation

$$y^2 - \frac{dy}{dx} = x^2 \frac{dy}{dx}$$ 2

(*ii*) A particle moves under the law $x = \text{Sin } k\,t$, where x is displacement at time t and k is constant. Show that the motion is simple harmonic. 2

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K. Sitaraman
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With Differential Equations

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